Business Mathematics with Calculus

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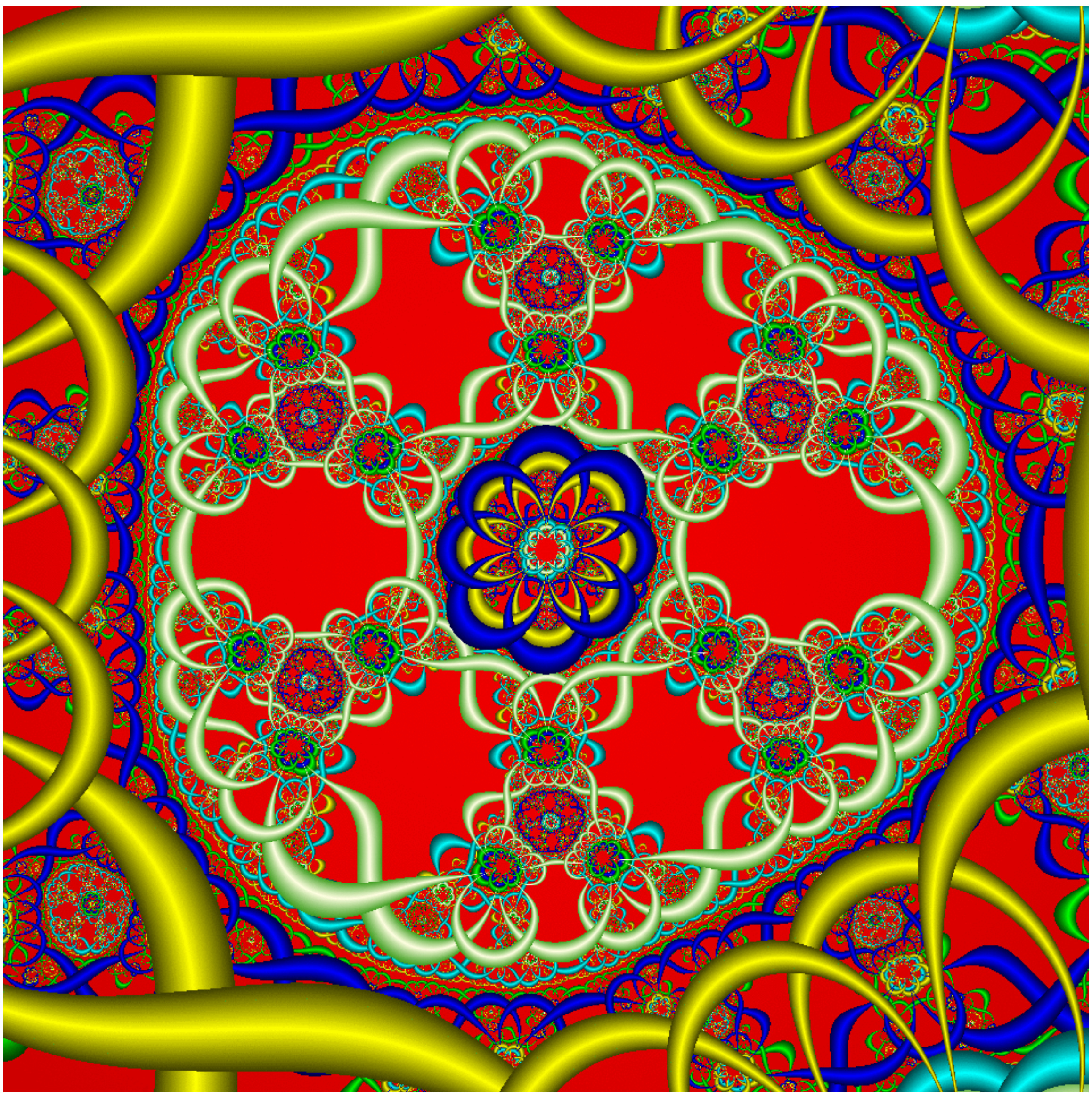
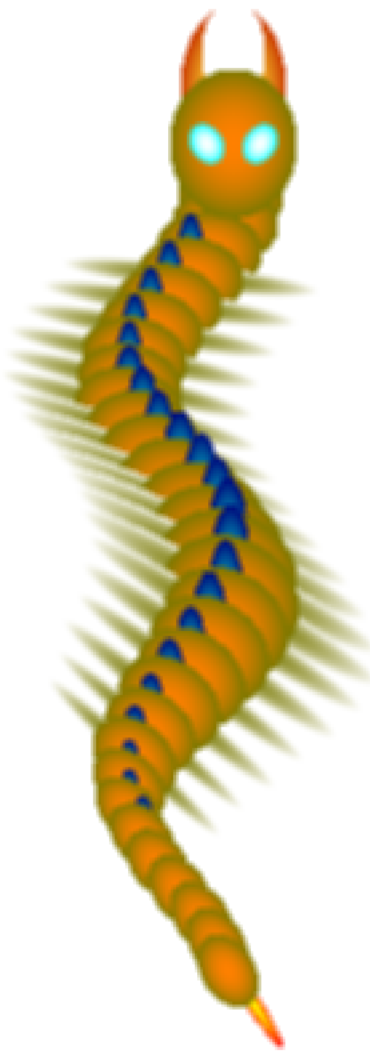


Beware of the Math!



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Chapter 1

Fundamentals

Let None But Geometers Enter Here
-inscribed about the entrance to Plato's Academy.

A student who is using these lecture notes is not likely to be a geometer (person who studies geometry) but is also unlikely to pass through the arch with the quotation on it. The original Academy was Plato's school of philosophy. It was founded approximately 25 centuries ago, in 385 BC at Akademia, a sanctuary of Athena, the goddess of wisdom and skill. Plato's motives for making this inscription are not recorded but he clearly felt that an educated person needed to know mathematics.

When am I going to use this crap?

-a typical exclamation from a student who is not putting in the hours needed to pass his one required math class.

The answer to the question above may well be "never". That doesn't mean the person who asked the question wouldn't benefit from basic mathematics. They could have benefitted from math, and its sister *quantitative reasoning*, but have chosen not to. There are only a few gainful activities where math is not present. The innumerate (this is the mathematical analog to illiterate) typically don't *notice* that their disability is harmful. They do get cheated, lied to successfully, and ripped off more often than other people. They also *don't* get promoted as often or paid as much. Mathematical skill also acts as a leveler between the sexes.

Although women earn significantly lower wages than men do across all levels of education and occupational categories, the gender wage gap is not significant among professional men and women with above-average mathematics skills. One way of reducing the gender wage gap would be to encourage girls to invest more in high school mathematics courses in order to improve their quantitative skills.

-Aparna Mitra, Mathematics skill and male-female wages *in* The Journal of Socio-Economics Volume m 31, Issue 5, 2002, Pages 443-456

Here endeth the sermon. This course is designed for a mix of students and skill levels. It assumes that many of the people in the course could be better prepared and may have an aversion to mathematics.

RULES FOR SURVIVAL

1. *Show up to class every single day.*
2. *Keep up with the material: do the readings, get the quizzes in on time.*
3. *Study with other people. Check one another's work, help one another.*
4. *Stick to the truth and there is good hope of mercy.*

1.1 Basic Algebra

The origin of the word *algebra* is the Arabic word "al-jabr" which means (roughly) "reunion". It is the science of reworking statements about equality so that they are more useful. We start with a modest example.

Example 1.1 *In this example we solve a simple one-variable equation.*

$3x + 7 = 16$ *This is the original statement.*

$3x + 7 - 7 = 16 - 7$ *Subtract seven from each side of the equation.*

$3x = 9$ *Resolve the arithmetic.*

$\frac{3x}{3} = \frac{9}{3}$ *Divide both sides of the equation by three.*

$x = 3$ *Resolve the arithmetic.*

Since the final statement contains a simpler and more direct statement about the value of x we judge it more useful. While the above example is almost insultingly simple in both its content and level of detail it introduces two important points.

- Algebra can take an equation all over the place. It is *your* job to steer the process to somewhere useful.
- Any algebraic manipulation consists of an application of one of a small number of rules to change an equation. Even if you know your exact or approximate destination (e.g.: solve for x) there is strategy that can be used to find a *short* (easier) path to that destination.

In example 1.1, subtracting 7 from both sides reduced the number of terms in the equation. Dividing by three finished isolating x. In both cases the steps clearly led toward the goal "solve for x".

1.1.1 Some Available Algebra Steps

The following are legal moves when applied to an equation. Some of them involve equations like log and inverse log (exponentials) that we will get to later in the chapter.

1. You may add (subtract) the same quantity to (from) both sides of the equation.

2. You may multiply both sides of the equation by the same quantity.

3. You may divide both sides of the equation by the same quantity **but only when the quantity is not zero**. Some of you may wonder how a quantity can sometimes be zero - this only happens if it contains a variable, like x.

4. You may square both sides of the equation.

5. You may take the square root of both sides of the equation **but only when the sides of the equation are at least zero**.

6. You may take the *log* or *ln* of both sides of the equation **but only when the sides of the equation are positive**.

7. You may take the inverse log of both sides of the equation.

8. You may cancel a factor from the top and bottom of a fraction. A *factor* is a part of an expression that is multiplied by the rest of the expression. In $2x + 2y = 2(x + y)$ 2 is a factor but, for example, $2x$ is not.

9. You may multiply a new factor into the top and bottom of a fraction.

There are other steps, and we will get to them later. The rules use the term "quantity" a lot. A quantity can be a number; it was 3 and 7 in Example 1.1, but it also can be an expression involving variables. The next example demonstrates this possibility. Both $(x - 2)$ and $(y - 1)$ appear as "quantities" in Example 1.2.

Example 1.2 *If* $y = \frac{x+1}{x-2}$ *solve the expression for* x.

$y = \frac{x+1}{x-2}$	*This is the original statement*
$y(x-2) = (x-2)\frac{x+1}{x-2}$	*The fraction is annoying, get rid of it by multiplying both sides by* $(x-2)$
$y(x-2) = \cancel{(x-2)}\frac{x+1}{\cancel{x-2}}$	*Cancel matching terms on the top and bottom of the fraction.*
$y(x-2) = x+1$	*Resolve the arithmetic.*
$yx - 2y = x+1$	*Distribute the* y *over* $(x-2)$.
$xy - 2y = x+1$	*Use the commutative law to put* x *and* y *in the usual order.*

-at this point we want all variables x on one side and everything else on the other-

$xy - 2y + 2y = x + 1 + 2y$	*Add* $2y$ *to both sides.*
$xy = x + 1 + 2y$	*Resolve the arithmetic.*
$xy - x = x - x + 1 + 2y$	*Subtract* x *from both sides.*
$xy - x = 1 + 2y$	*Resolve the arithmetic.*
$x(y-1) = 1 + 2y$	*Factor* x *out from the terms on the left hand side.*
$\frac{x(y-1)}{y-1} = \frac{1+2y}{y-1}$	*Cancel matching terms on the top and bottom of the fraction.*
$\frac{x\cancel{(y-1)}}{\cancel{y-1}} = \frac{1+2y}{y-1}$	*divide both sides by* $(y-1)$.
$x = \frac{1+2y}{y-1}$	*Resolve the arithmetic; we have* x *and are done.*

Example 1.2 is done one **tiny** step at a time. One of the things we will learn is more efficient steps that let us do algebra in fewer steps. The small step size in early examples is intended to provide clarity for those who haven't had a math course in a while. A potential bad side effect of this stepwise clarity is that it can completely obscure the strategy for actually solving the problem. It is possible to understand all the steps but miss the point of the problem. Keep this unfortunate duality in mind during the early steps and try to see both the strategy and tactics for solving the problem. We will make an effort to show you how to run algebra faster in later parts of this chapter.

Our next example will show us how to solve for x when there is a square root in the way. As with the fraction that we eliminated first in Example 1.2, the square root will be the most annoying part of the problem and so should be eliminated first, if possible.

Example 1.3 *If $\sqrt{x+1} = 2$ solve the expression for x.*

$\sqrt{x+1} = 2$	*This is the original statement.*
$\sqrt{x+1} \times \sqrt{x+1} = 2 \times 2$	*Square both sides of the equation to get rid of the square root.*
$x + 1 = 4$	*Resolve the arithmetic. Remember that $\sqrt{Bob} \times \sqrt{Bob} = Bob$ and don't actually multiply anything out here.*
$x + 1 - 1 = 4 - 1$	*Subtract one from both sides of the equation.*
$x = 3$	*Resolve the arithmetic, and we are done.*

1.1.2 Order of Operations

The statement $3 \times x + 4 \times y^2$ means you should execute the following steps in the following order.

1. Square y,
2. multiply x by three,
3. multiply the result of squaring y by 4,
4. add the results of steps 2 and 3, to obtain the final answer.

The troubling part of this is that the operations are *not* hitting in the normal left-to-right reading order. This is because of **operator precedence**. An *operator* is something that can change a number or combine two numbers. Example 1.3 in the above computation are squaring, multiplying, and adding. *Operator precedence* is the convention that some operators are more important and hence are done first. If there were no such rules we could give the order in which we want things done with parenthesis (things inside parentheses are always done first) by saying:

$$((3 \times x) + (4 \times (y^2)))$$

but that looks ugly and uses a lot more ink. Here are some of the operator precedence rules.

1. Anything enclosed in parentheses is done first (has the highest precedence).
2. Minus signs that mean something is negative come next; these are different from minus signs that mean subtraction. E.g. -2 means "negative 2" not "something is getting 2 subtracted from it".
3. Exponents come next. Remember that $\sqrt{x} = x^{\frac{1}{2}}$ so roots have the same precedence as exponents.
4. Multiplication and division come next with one exception for division, explained below.
5. Addition and subtraction come next.
6. Things with the same precedence are executed left to right. Usually this doesn't matter because of facts like 1+(2+3)=6=(1+2)+3 which make the order irrelevant.

The **exception for division** concerns the long division bar. The expression

$$\frac{x+1}{2x-1}$$

means $(x+1)/(2x-1)$. The top and bottom of a division bar have implicit (invisible) parenthesis.

1.1.3 Fast Examples

Following up on the remark that detailed steps can obscure overall solution methods, we are now going to repeat earlier examples, using faster steps with terser descriptions.

Example 1.4 Problem: *solve* $3x + 7 = 16$ *for* x.
Solution:

$3x + 7 = 16$	*This is the original statement.*
$3x = 9$	*Subtract 7 from both sides.*
$x = 3$	*Divide both sides by 3. Done.*

Example 1.5 Problem: *solve* $y = \frac{x+1}{x-2}$ *for* x.
Solution:

$y = \frac{x+1}{x-2}$	*This is the original statement.*
$yx - 2y = x + 1$	*Clear the fraction and distribute* y.
$yx - x = 2y + 1$	*Get all terms with an* x *one one side, everything else on the other.*
$x(y - 1) = 2y + 1$	*Factor the left hand side to get* x *by itself.*
$x = \frac{2y+1}{y-1}$	*Divide both sides by* $y - 1$. *Done.*

Example 1.6 Problem: *solve* $\sqrt{x+1} = 2$ *for* x.
Solution:

$\sqrt{x+1} = 2$	*This is the original statement.*
$x + 1 = 4$	*Square both sides.*
$x = 3$	*Subtract 1 from both sides. Done.*

1.1.4 Sliding for multiplication and division.

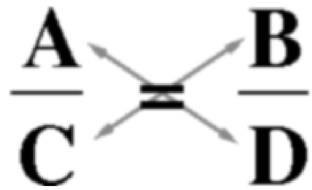
If we have the equation

$$\frac{A}{C} = \frac{B}{D}$$

Then multiplying or dividing by any of the expressions A, B, C, or D can be thought of as sliding them along diagonals through the equals sign. Applying this sliding rule one or more times permits us to solve for each of the four expressions:

$$A = \frac{BC}{D} \quad B = \frac{AD}{C} \quad C = \frac{AD}{B} \quad \text{and} \quad D = \frac{BC}{A}$$

Notice that we reversed the direction of the equality to always place the single variable on the left. This is standard practice. The following diagram shows how terms in an equality may slide.



Now let's look at a more complicated equation.

$$\frac{A \times B}{C + D} = \frac{x + y}{Q \times R}$$

If we multiply both sides by R we get

$$\frac{A \times B \times R}{C + D} = \frac{x + y}{Q}$$

If, instead, we divide both sides by $(x + y)$ we would obtain

$$\frac{A \times B}{(x + y) \times (C + D)} = \frac{1}{Q \times R}$$

If we think of the terms be multiplied or divided by as *sliding* along diagonals of the short shown in the diagram, then we can rapidly rearrange an equation of this sort. **Warning:** Notice that this technique only works if the expressions are parts of multiplied groups items. We can slide $(x + y)$ as a single object but we *cannot* slide x or y individually; this is because "divide both sides by x (or y)" wold not correctly cancel the term $(x + y)$ under the older, slower rules.

Example 1.7 Problem: *Using the technique of sliding, solve* $\frac{A \times B}{C+D} = \frac{x+y}{Q \times R}$ *for A, B, Q, and R.*

Solutions:

$$A = \frac{(x + y)(C + D)}{Q \times R \times B}$$

$$B = \frac{(x + y)(C + D)}{Q \times R \times A}$$

The next two are harder because the target variable is on bottom but if you keep sliding terms along diagonals you get:

$$Q = \frac{(C + D)(x + y)}{A \times B \times R}$$

$$R = \frac{(C + D)(x + y)}{A \times B \times Q}$$

The next example shows how to solve for a term that is not just multiplied by the others.

Example 1.8 Problem: *Using the technique of sliding, solve* $\frac{A \times B}{C+D} = \frac{x+y}{Q \times R}$ *for x.*

Solution:
Start by sliding to obtain

$$(x + y) = \frac{A \times B \times Q \times R}{C + D}$$

Now subtract y from both sides and get

$$x = \frac{A \times B \times Q \times R}{C + D} - y \tag{1.1}$$

For some purposes this may not be in simplest form because the right hand side is not a single, large fraction. We will deal with this in the section on fractions.

1.1.5 Arithmetic of Fractions

Normally multiplication seems more difficult than addition but, when dealing with fractions, this usual state of affairs is reversed. In this section we review the arithmetic of fractions. The first notion is that of **putting fractions in reduced form**. The following statement is true:

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \frac{-2}{-6} = \frac{-5}{-15}$$

and shows that the fraction we call "one third" can be written a lot of different ways. When answering a problem we always put a fraction into the form so that the top and bottom have no common factors. Thus the *reduced form* of $\frac{4}{12}$ is $\frac{1}{3}$.

This rule also applies when variables are involved. So, for example, the reduced form of $\frac{x}{3x}$ is $\frac{1}{3}$. A variable is only eliminated if it can be factored out of every term in the top and bottom.

Example 1.9 Fractions and their reduced forms:

Fraction	*Reduced form*	*Comment*
$\frac{4}{36}$	$\frac{1}{9}$	*Common factor is 4.*
$\frac{132}{15}$	$\frac{44}{5}$ *or* $8\frac{4}{5}$	*Common factor is 3.*
$\frac{91}{63}$	$\frac{13}{9}$ *or* $1\frac{4}{9}$	*Common factor is 7.*
$\frac{-6}{-54}$	$\frac{1}{9}$	*Common factor is -6.*
$\frac{4}{-10}$	$-\frac{2}{5}$	*Common factor is -2; minus signs should end up on top or out front, not on the bottom.*
$\frac{x}{3xy}$	$\frac{1}{3y}$	*Common factor is x.*
$\frac{xy+y^2}{2x+2y}$	$\frac{y}{2}$	*Common factor is $x+y$; note: $\frac{xy+y^2}{2x+2y} = \frac{y(x+y)}{2(x+y)}$*

Basic Arithmetic for Fractions

A fraction $\frac{n}{d}$ is made by dividing two expressions the *numerator* n and the *denominator* d. Multiplying fractions is easy, you just multiply the numerators and denominators:

$$\frac{1}{2} \times \frac{3}{5} = \frac{1 \times 3}{2 \times 5} = \frac{3}{10}$$

or if the expressions making up the fraction are variables

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

Note that in the latter example we are shortening $a \times c$ to ac and $b \times d$ to bd. This is a standard, alternate method of denoting multiplication. It takes less space and we will use this alternate notation frequently from now on.

You may add two fractions **only if they have the same denominator**. This means that if two fractions do not already have a common denominator, you need to modify them so they have one. If fractions already have the same denominator you simply add the numerators. For example, adding seven halves and four halves yields seven plus four halves or eleven halves:

$$\frac{7}{2} + \frac{4}{2} = \frac{7+4}{2} = \frac{11}{2}$$

Example 1.10 Problem: *Compute* $\frac{1}{4} + \frac{5}{7}$.

Solution: *The smallest number that is a multiple of both 4 and 7 is 28. Recall that we may multiply the top and bottom of a fraction by the same number without changing its value.*

$$\begin{aligned}\frac{1}{4} + \frac{5}{7} &= \frac{1}{4} \times \frac{7}{7} + \frac{5}{7} \times \frac{4}{4} \\ &= \frac{7}{28} + \frac{20}{28} \\ &= \frac{27}{28}\end{aligned}$$

To correct each fraction to the common denominator we simply multiply by the missing factor divided by itself. The ability to factor numbers is an important part of figuring out what the common denominator is.

The common denominator is also needed when the expressions in the fractions have variables in them.

Example 1.11 Problem: *Compute* $\frac{x}{y} + \frac{y}{x+y}$.

Solution: *The simplest expression that is a multiple of both x and $(x+y)$ is $x(x+y)$. Recall that we may multiply the top and bottom of a fraction by an expression without changing its value.*

$$\begin{aligned}\frac{x}{y} + \frac{y}{x+y} &= \frac{x}{y} \times \frac{x+y}{x+y} + \frac{y}{x+y} \times \frac{y}{y} \\ &= \frac{x(x+y)}{y(x+y)} + \frac{y \times y}{y(x+y)} \\ &= \frac{x^2+xy}{xy+y^2} + \frac{y^2}{xy+y^2} \\ &= \frac{x^2+xy+y^2}{xy+y^2}\end{aligned}$$

Reciprocals of fractions, dividing fractions.

The reciprocal of a number n is the number one divide by n so, for example, the reciprocal of 2 is $\frac{1}{2}$. In order to take the reciprocal of a fraction you interchange the numerator and denominator (flip the fraction over). So:

$$\frac{1}{2/3} = \frac{3}{2}$$

Since dividing by something is equivalent to multiplying by its reciprocal, this gives us an easy rule for dividing to fractions: flip the one you're dividing by over and multiply instead.

$$\frac{1/3}{1/5} = \frac{1}{3} \times \frac{5}{1} = \frac{5}{3}$$

These rules apply to expressions involving variables as well. This means that

$$\frac{1}{\frac{x+y}{x-y}} = \frac{x-y}{x+y}$$

for example. If two fractions are divided then one multiplies by the reciprocal of the fraction forming the denominator. This is called the **invert and multiply** rule for dividing fractions. Symbolically:

$$\frac{a/b}{c/d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

We will use the phrase *invert and multiply* for this method of resolving the division of fractions from this point on in the notes. The following example shows a division of fractions consisting of expressions involving variables. In it the step "Resolve the binomial multiplications with the distributive law" appears. This step actually uses the distributive law twice:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$$

Example 1.12 Problem: *Simplify the expression* $\frac{\frac{x+1}{y+1}}{\frac{13}{x-y}}$.

Solution:

$\frac{\frac{x+1}{y+1}}{\frac{13}{x-y}}$	*This is the original problem.*
$\frac{x+1}{y+1} \times \frac{x-y}{13}$	*Invert and multiply.*
$\frac{(x+1)(x-y)}{(y+1)\times 13}$	*Multiply the numerators and denominators.*
$\frac{x^2-xy+x-y}{13y+13}$	*Resolve the binomial multiplications with the distributive law. Done.*

Exercises

Exercise 1.1 *Solve each of the following expressions for the stated symbol.*

a) $3x + 2 = 11$ *for* x.

b) $3y - 2x = y + 7x + 2$ *for* y.

c) $xy + 1 = 2x + 2$ *for* x.

d) $\sqrt{2x+1} = 3$ *for* x.

e) $(y+1)^3 = 27$ *for* y *(Hint: take a third root).*

f) $(x+y+1)^2 = 16$ *for* x *(Remember the* $\pm$ *on the square-root).*

g) $y = \frac{x+2}{x-3}$ *for* x.

h) $3ab - 3cd = 0$ *for* a.

i) $\frac{y-1}{x+1} = 2$ *for* x.

j) $\frac{y-1}{x+1} = 2$ *for* y.

Exercise 1.2 *What is the value, rounded to three decimals, of the expression*

$$\frac{\sqrt{x^2+1}+3x}{x^2+3x+4}$$

when x =0, 1, and 2? Give three answers.

Exercise 1.3 *What is the value, rounded to three decimals, of the expression*

$$\sqrt[3]{\frac{x+2}{x-2}}$$

when x =0, 1, and -1?

Exercise 1.4 *What is the value, rounded to three decimals, of the expression*

$$\frac{x^3+3x^2+3x+1}{x^2+2x+1}$$

when x =3, 4, and 5?

Exercise 1.5 *What is the value, rounded to three decimals, of the expression*

$$\frac{\left(\sqrt{x-1}+2\right)\left(\sqrt{x-1}-2\right)}{1-x}$$

when x =2, 3, and 4?

Exercise 1.6 *For the expression* $(3x+2y^3)^2$ *state in English phrases the operations in the order they occur. An example of this sort of exercise appears at the beginning of section 1.1.2*

Exercise 1.7 *For the expression* $3\sqrt{x^2+1}+7$ *state in English phrases the operations in the order they occur. An example of this appears at the beginning of section 1.1.2*

Exercise 1.8 *Using the technique of sliding, and any other algebra required, solve each of the following for every variable (letter) in the expression.*

a) $\frac{xy}{rs} = 6$. **b)** $\frac{(a+b)c}{uv} = 1$. **c)** $\frac{1}{a} = \frac{cd}{s}$. **d)** $\frac{x+y}{a} = \frac{u+v}{t}$. **e)** $\frac{abc}{d} = \frac{x}{y}$. **f)** $\frac{(a+b)(c+d)}{2} = uv$.

Exercise 1.9 *Reduce the following fractions to simplest form. Also report the common factor. So, for example, the answer for 8/12 would be "2/3, the common factor is 4".*

a) $\frac{255}{40}$. **b)** $\frac{-255}{51}$. **c)** $\frac{255}{27}$. **d)** $\frac{120}{84}$. **e)** $\frac{125}{625}$. **f)** $\frac{x^2}{3xy}$. **g)** $\frac{9y^2}{3xy}$. **h)** $\frac{x^2-4x+4}{x^2-3x+2}$. **i)** $\frac{abc}{abd+abe}$. **j)** $\frac{255x^2+17x}{34x}$.

Exercise 1.10 *Compute the following expressions on fractions, placing the results in simplest form.*

a) $1/2+1/3$. **b)** $1/2-1/3$. **c)** $3/4+5/7$. **d)** $255/34-255/51$. **e)** $91/14-1/2$. **f)** $\frac{1}{x}-\frac{1}{y}$.

g) $2x+\frac{1}{x}$. **h)** $\frac{x}{y}-\frac{2}{x}$. **i)** $\frac{1}{x+h}-\frac{1}{x}$. **j)** $\frac{1}{2}+\frac{1}{3}+xy$.

Exercise 1.11 *Compute the following expressions on fractions, simplifying and placing the results in simplest form. Be sure to reduce the result to a single fraction.*

a) $\frac{1}{3}\times\frac{5}{8}-\frac{1}{2}$. **b)** $\frac{1}{4}\div\frac{2}{x}$. **c)** $\frac{x}{3}\div 2y-\frac{2}{3}x$. **d)** $\frac{1}{n}-\frac{1}{n+1}$. **e)** $\frac{1}{n}\times 1n+2+\frac{2}{n+1}$. **f)** $\frac{1}{2}x\div\frac{3}{y}+1$.

g) $\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}}$. **h)** $\frac{\frac{1}{x}-\frac{1}{y}}{xy}$. **i)** $\frac{(x+y)(x-y)}{\frac{1}{x}+2}$. **j)** $\frac{(x+h)^2-x^2}{h}$.

Exercise 1.12 *Suppose that an expression is the ratio of one more than the square of x and two minus the square of y. Write the expression in algebraic notation.*

Exercise 1.13 *If the expression in problem 1.12 is equal to one, solve it for both x and y.*

Exercise 1.14 *Write an algebraic expression for the following quantity. The third power of the sum of twice x and three times y.*

Exercise 1.15 *Write an algebraic expression for the following quantity. Two more than the square root of one more than the ratio of a to b.*

Exercise 1.16 *Suppose that the cost of manufacturing n units of a widget includes a \$1200 setup charge, uses \$18.42 of parts for each widget, uses \$0.88 of labor to assemble each widget, and has a charge of \$0.07 per widget for the amount the factory wears out making the widget. Write an expression for the marginal cost of making widgets that depends on the number of widgets made.*

Exercise 1.17 *Suppose we add up the numbers $\frac{1}{n}$ for $n=2,3,\ldots,7$. What is the common denominator of the result?*

Exercise 1.18 *Suppose that in a partnership general partners divide one-third of the profits, tier one partners divide one-half the profits, and tier two partners divide the remainder of the profits. If there are four general partners, twenty tier one partners, and one hundred and sixty tier two partners then what fraction of the profits does each sort of partner get?*

Exercise 1.19 *Suppose that we are dividing a pie in the following odd fashion. Each person, in order, gets one-quarter of the remaining pie until the amount of pie left is one-quarter of the pie or less. This last piece of pie is given to the last person. What fraction of the pie is the smallest piece of pie handed out?*

Exercise 1.20 *Suppose that two people are supposed to divide a small cake so that each person feels they have gotten al least their fair share. The cake has decorations on it that are different in different places so the two people may have different opinions about how good a given piece is. Give a method for dividing the cake and demonstrate logically that each person will feel they have gotten at least their share.*

1.2 Lines and Quadratic Equations

In this section we review formulas that use the first power of the variable (lines) and those that use the second (quadratics).

1.2.1 Equations of Lines

Lines are equations in which there are two variables both of which are raised to the first power. Here are some examples of lines:

$$y = 3x + 1$$

$$2x + 4y = 7$$

$$2(x - 1) + 3(y - 5) = -1$$

Notice that these lines are all in different forms. The first one considered to be simplified, the other two forms may be useful for some other reason. There are two forms we will often use for lines: **slope intercept** and **point slope**. The slope-intercept form of a line is

$$y = mx + b$$

where m is the **slope** of the line and b is the **intercept** or y-**intercept** of the line. The *slope* is the steepness of the line going from left to right. A line with slope m increases in the y direction by m units whenever x increases by one unit. The *intercept* is the value of the line when $x = 0$ or, alternatively, the value on the y-axis where the line hits the axis. The x-intercept is the value of x when $y = 0$ - the value of x when the line hits the x axis. Figure 1.1 shows an example.

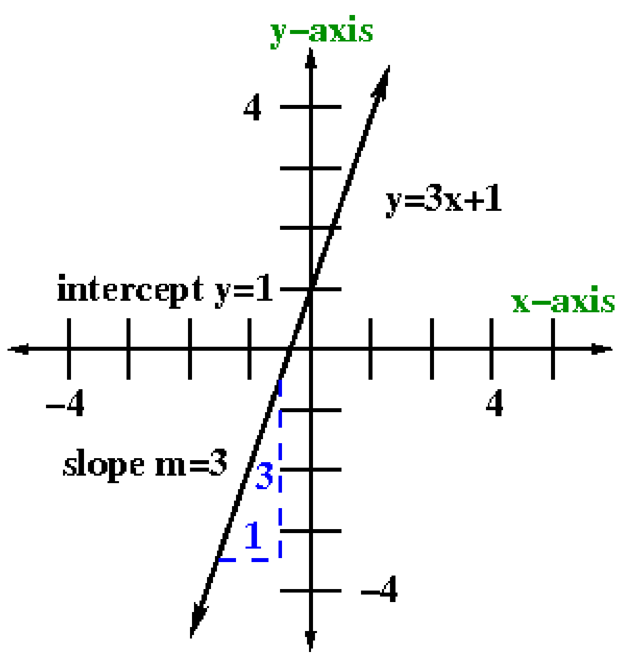


Figure 1.1: The graph of the line $y = 3x + 1$ showing the slope and intercept.

The point-slope form of a line is most often used to construct a line with a slope m going through a point (a, b). It has the form:

$$(y - b) = m(x - a)$$

If we plug the points $x = a$, $y = b$ into this formula we get

$$(b - b) = m(a - a)$$

$$0 = m \cdot 0$$

$$0 = 0$$

which is a true statement, so the point (a, b) is on the line. It is possible to convert a line in point-slope form into one in slope-intercept form:

$$\begin{aligned} (y - b) &= m(x - a) \\ y - b &= mx - ma \\ y &= mx - ma + b \end{aligned}$$

This demonstrates that the line does have slope m and that the intercept is equal to $(-ma + b)$.

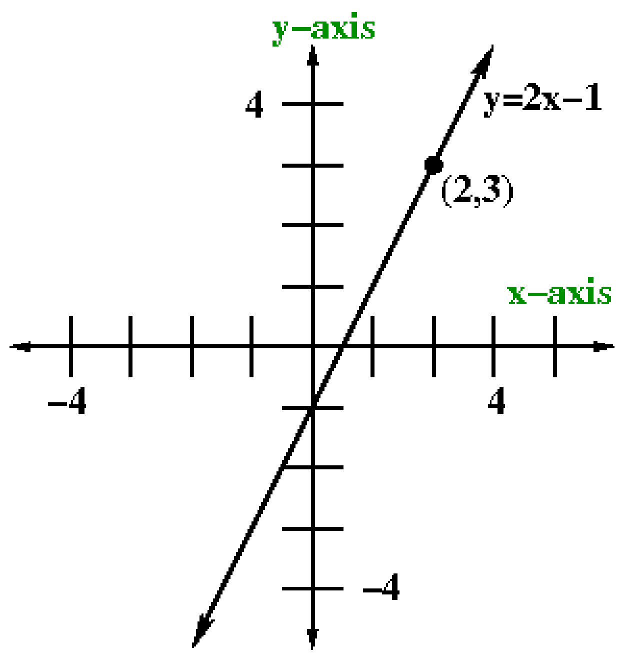


Figure 1.2: The line $(y - 3) = 2(x - 2)$ (also $y = 2x - 1$) with the point (2,3) displayed.

Example 1.13 Using the point slope form

Problem: *Construct a line of slope 2 that contains the point $(2, 3)$. Place the line in slope-intercept form. Start with the point-slope form and then plug in the desired point and slope.*

$$\begin{aligned} (y - b) &= m(x - a) \\ (y - 3) &= 2(x - 2) \\ y - 3 &= 2x - 4 \\ y &= 2x - 4 + 3 \\ y &= 2x - 1 \end{aligned}$$

Figure 1.2 shows the resulting line and the point (2,3). An important thing to remember is that a line has a single, unique slope-intercept form but it has a different point-slope form for every one of the infinitely many points on the line. This means that if we are comparing lines to see if they are the same it is necessary to place the lines in slope-intercept form.

Two points determine a line: which one?

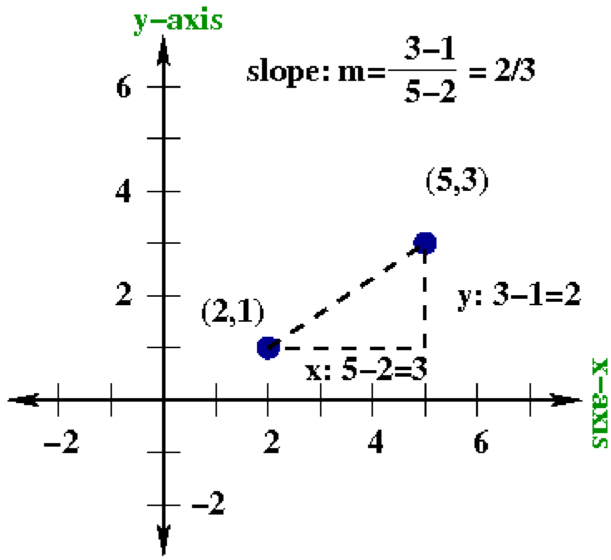


Figure 1.3: Details of finding that a line defined by the points (2,1) and (5,3) has a slope of 2/3.

You have probably heard the saying that "two points determine a line". So far we can find a line from one point and a slope, using the point-slope formula. If we have two points and want to know the equation of the line containing both of them, the easy method is to find the slope and then apply the point-slope formula to either one of the points. Recall that slope is the amount y increases when x increases by one. We cold also say that the slope of the line is its *rise* over its *run*. In this case rise is the distance the line moves in the y direction while run is the distance it moves in the x direction. If we have two points we can simply divide the y distance between the points by the x distance between the points and get the slope of the line between them. Figure 1.3 illustrates the process.

Example 1.14 Finding a line through two points.
Problem: *find the equation in slope-intercept form of the line through the points (2,1) and (5,3).*

This example uses the picture in Figure 1.3. The rise from 1 to 3 is 3-1=2; the run from 2 to 5 is 5-2=3. Computing slope as rise over run we get $m = \frac{rise}{run} = 23$ *and the slope of the line is* $m = 2/3$. *We can now use the point slope formula with either of the points - since (2,1) has smaller coordinates we will, somewhat arbitrarily, choose this point. This gives us:*

$$(y-1) = \frac{2}{3}(x-2)$$

$$y-1 = \frac{2}{3}x - \frac{4}{3}$$

$$y = \frac{2}{3}x - \frac{4}{3} + 1$$

$$y = \frac{2}{3}x - \frac{1}{3}$$

The final answer, in slope-intercept form, is $y = (2/3)x - (1/3)$.

In the next example we give several examples of finding the slope of the line between pairs of points.

Example 1.15 Slopes derived from pairs of points

Points	*Rise*	*Run*	*Slope*
(1,1);(3,5)	*5-1=4*	*3-1=2*	$m = \frac{5-1}{3-1} = \frac{4}{2} = 2$
(2,-2);(3,3)	*3-(-2)=5*	*3-2=1*	$m = \frac{3-(-2)}{3-2} = \frac{3+2}{3-2} = 5$
(-1,-1);(4,2)	*2-(-1)=3*	*4-(-1)=5*	$m = \frac{3}{5}$
(-1,-2);(-3,-1)	*-1-(-2)=1*	*-3-(-1)=-2*	$m = \frac{1}{-2} = -\frac{1}{2}$
(1,1);(a,b)	*b-1*	*a-1*	$m = \frac{b-1}{a-1}$

Earlier we said that the slope-intercept form of a line is unique but a line may have many point-slope forms. By plugging $x = 1$ and $x = 3$ into the line $y = 2x - 1$ we can find that the points (1,1) and (3,5) are both on the line. The next example shows that the slope-intercept form of the line for each of these points is the same.

Example 1.16 Different point-slope forms: sample slope-intercept
Problem:*Find the line of slope 2 through each of the points (1,1) and (3,5).*

Compare the slope-intercept forms of these lines.

First the point (1,1):

$$\begin{aligned} (y-1) &= 2(x-1) \\ y-1 &= 2x-2 \\ y &= 2x-2+1 \\ y &= 2x-1 \end{aligned}$$

Now the point (3,5):

$$\begin{aligned} (y-5) &= 2(x-3) \\ y-5 &= 2x-6 \\ y &= 2x-6+5 \\ y &= 2x-1 \end{aligned}$$

As expected, the slope-intercept forms are identical indicating that both point-slope forms are different equations for the same line.

The next step in our discussion of equations of lines is giving a formula for the slope a line through two points in terms of the coordinates of those points and exploring a special slope that may cause a problem.

Formula 1.1 Slope of a line through two points *If we have two points* (x_1, y_1) *and* (x_2, y_2) *then the slope of the line though those points is either*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

or, if $x_1 = x_2$ *the slope* does not exist. *A line whose slope does not exist is a* vertical line. *Its rise over run involves dividing by zero, which is what causes the problem.*

Vertical lines are given by a formula of the kind $x = c$ for some constant c. They consist of all points (c, y) where y can take on any value. The slope of vertical lines is said to be *undefined*. It can be thought of, informally, as being infinite but this is not a well defined notion and should only be used informally (i.e. in discussion but *not* on an examination).

Parallel and Right-Angle Lines

Once we know how to find the slopes of lines, a very simple rule lets us determine when two lines are parallel or intersect one another at right angles.

Fact 1.1 *Two lines are parallel if and only if they have the same slope.*

Example 1.17 Problem: *are any two of the following three lines parallel?*

L1: $y = 2x + 1$

L2: $3y - 6x = 7$

L3: $3x + y = 3$

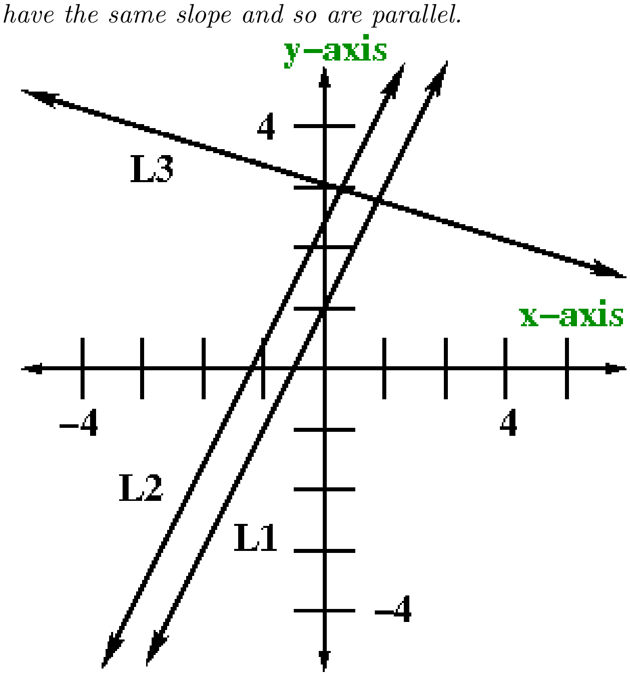
Notice that the slope of L2 and L3 are not obvious because they are not in a form that explicitly displays their slope. Placing the lines into slope-intercept form:

L1: $y = 2x + 1$ *(already in SI-form, included for completeness).*

L2: $y = 2x + \frac{7}{3}$

L3: $y = -3x + 3$

And we see that L1 and L2 have the same slope and so are parallel.



Fact 1.2 *Two lines with slopes m_1 and m_2 intersect at right angles if and only if*

$$m_1 = -\frac{1}{m_2}$$

In other words if their slopes are negative reciprocals of one another.

Example 1.18 Problem: *Find a line that intersects $y = 2x - 1$ at right angles at the point (1,1).*

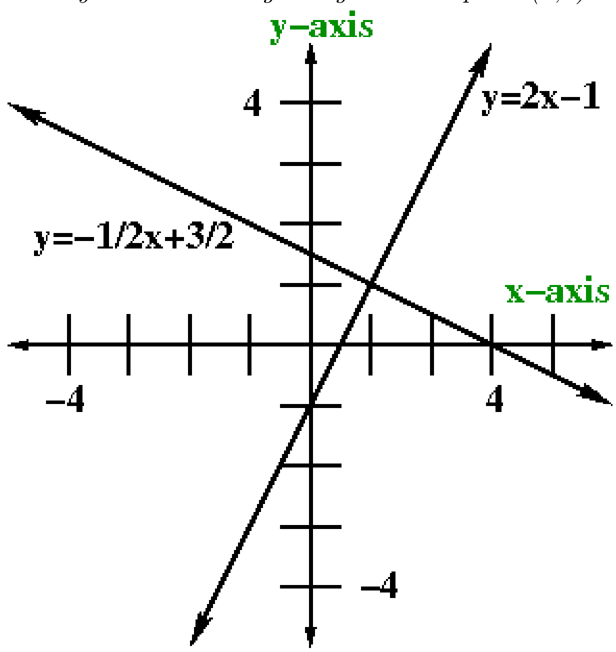
First of all, double check that the point (1,1) is one the line $y = 2x - 1$: $2 \times 1 - 1 = 1$ (check). The slope of the given line is $m_1 = 2$. A line intersecting it at right angles would, by the fact above, have a slope of

$$-\frac{1}{2}$$

We now have a point (1,1) and a slope $m = -1/2$ and so we can build a line with the point-slope formula.

$$\begin{aligned}
(y-1) &= -\frac{1}{2}(x-1) \\
y - 1 &= -\frac{1}{2}x + \frac{1}{2} \\
y &= -\frac{1}{2}x + \frac{1}{2} + 1 \\
y &= -\frac{1}{2}x + \frac{3}{2}
\end{aligned}$$

And we have the line that intersect $y = 2x - 1$ at right angles at the point (1,1).



Finding the Intersection of Lines

A fairly common situation is having two lines and wanting to find the points that are on both lines (equivalently: that make both equations true). A simple algorithm can do this:

Algorithm 1.1 Finding the intersection of lines

Step 1: *Place the lines in slope-intercept form.*
$y = m_1x + b_1$
$y = m_2x + b_2$

Step 2: *Since points on both lines have the same y coordinate:*
$y = y$, *so*
$m_1x + b_1 = m_2x + b_2$

Step 3: *Solve the equation for x.*

Step 4: *Plug x into either line to get y.*

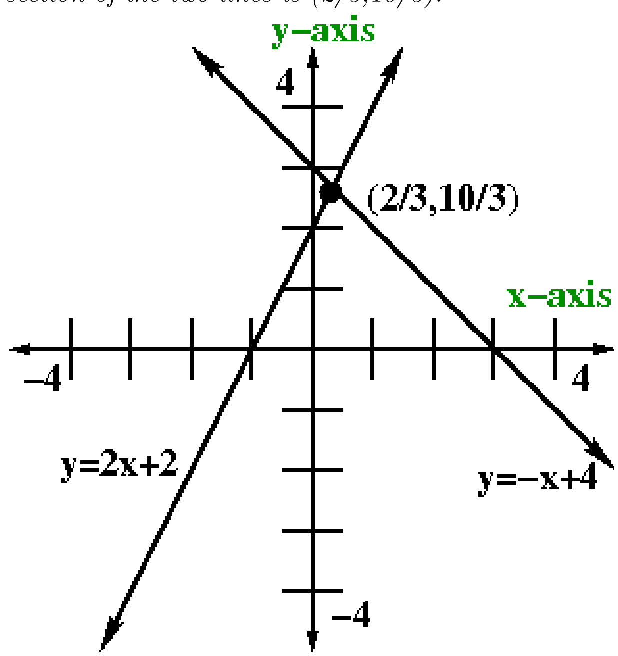
Step 5: *You have the point of intersection.*

Example 1.19 Problem: *find the intersection of $y = 2x + 2$ and $y = -x + 4$.*

This example is a bit of a softball because the lines are already in slope-intercept form and we get Step 1 for free. We start with Step 2:

$$y = y$$
$$2x + 2 = -x + 4$$
$$3x = 2$$
$$x = \frac{2}{3}$$

So now we know the point on both lines has $x = 2/3$. Plugging this into the second line we get $y = -2/3+4 = 10/3$. This means the point of intersection of the two lines is (2/3,10/3).

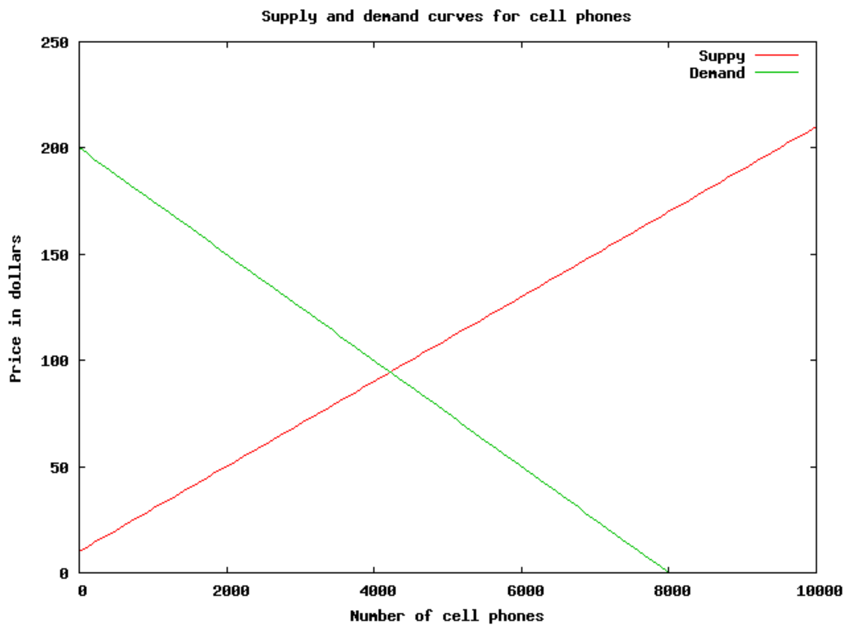


One important point: two lines that have the same slope don't intersect unless they are really the same line. This means that you can apply the algorithm and get no answer; typically if you plug parallel lines into the algorithm a divide by zero will happen.

Application: Balancing Supply and Demand

A **supply curve** tells us how many units of a commodity manufacturers will offer for sale at a given price. A **demand curve** tells us how many units of a commodity consumers will be willing to buy at a given price. As the price rises, manufacturers are willing to make more items but consumers are willing to purchase fewer. There is a **balance point** or **equilibrium** in which the number of units manufacturers are willing to supply and consumers are willing to purchase are equal.
We will use the variables p (for price) and q (for quantity) rather than the usual x and y. Suppose that the supply and demand curves for inexpensive cell phones are:

$$\text{Supply: } p = 10 + q/50 \qquad \text{Demand: } p = 200 - q/40$$



The graph of the supply and demand curves shows that the balance, where cell phones offered for sale and cell phones consumers are willing to purchase, is a little over 4000. Let's intersect the lines and find the exact value. This is Algorithm 1.1, just with different variable names.

$$p = p$$
$$10 + q/50 = 200 - q/40$$
$$q/50 + q/40 = 190$$
$$\frac{4q}{200} + \frac{5q}{200} = 190$$
$$\frac{9q}{200} = 190$$
$$q = \frac{38000}{9} \cong 4222$$

The symbol $\cong$ means "approximately" and is used because the answer is rounded to the nearest cell phone. If we plug that quantity into the supply curve we see the price at the balance point is

$$10 + 4222/50 = \$94.44$$

1.2.2 Solving Quadratic Equations

A **quadratic equation** is an equation like $y = x^2 + 3x + 2$ or $y = 4x^2 + 4x + 1$. The general form for a quadratic equations is $y = ax^2 + bx + c$ where a, b, and c are unknown constants. We insist that $a \neq 0$ so that the quadratic has an squared term, in other words a quadratic equation must have a squared term but may have no higher order terms.

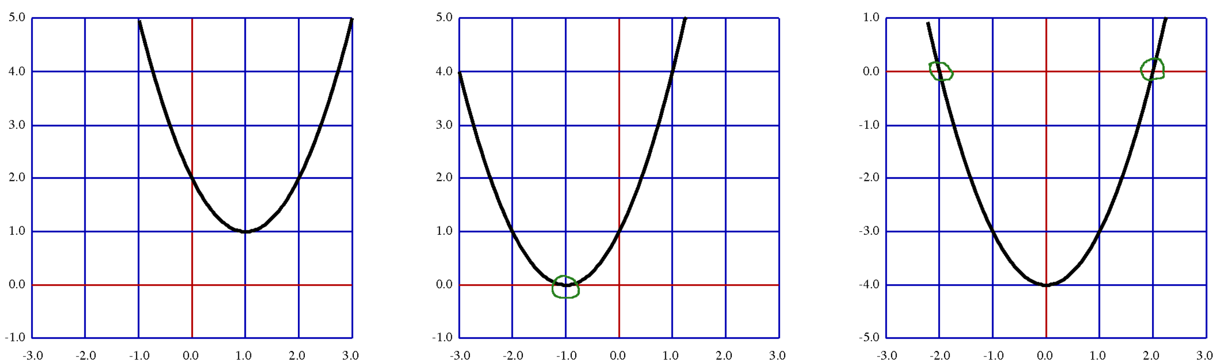
The **roots** of a quadratic equation are those values of x, or whatever the independent variable is, that make y, or whatever the dependent variable is zero. There are three methods for finding the roots of a quadratic equation:

1. Factoring,

2. completing the square, and

3. the quadratic equation.

It is also important to know that a quadratic equation may have zero, one, or two solutions. Figure 1.4 gives examples of all three of these possibilities. We will explore how to distinguish these three types of quadratics later.



x^2-2x+2 (no roots). x^2+2x+1 (one root). x^2-4 (two roots).

Figure 1.4: Quadratic equations with zero, one, or two roots. The roots are circled in green.

Factoring to Solve Quadratics

The first method of solving quadratic equations we will study is factoring.

Example 1.20 Problem: *find the roots of $y = x^2 + 3x + 2$ by factoring.*
Solution:

Notice that $2 \times 1 = 2$ but $1 + 2 = 3$. Since

$$(x + u)(x + v) = x^2 + ux + vx + uv = x^2 + (u + v)x + uv$$

these facts let us deduce that

$$x^2 + 3x + 2 = (x + 1)(x + 2)$$

It remains to find the values of x that make $y = 0$. With the factorization we can use the following rule:
If the product of two numbers is zero then one, the other, or both of those numbers are zero.

Compute:

$$
\begin{aligned}
x^2+3x+2 &= 0\\
(x+1)(x+2) &= \\
&SO\\
x+1 &= 0\\
x &= -1\\
&OR\\
x+2 &= 0\\
x &= -2
\end{aligned}
$$

and we say "$x = -1$ or -2" which can be written in shorthand as $x = -1, -2$.

In order to factor a quadratic you need to find two roots, like 1 and 2 in the example above, so that the numbers **add** to make the coefficient of x and **multiply** to make the constant term. You also need to experiment with the signs ($\pm$) of the roots to make the signs of the numbers in the quadratic come out correctly.

Completing the Square

In order to complete the square we need to understand what it means for a quadratic to be a **perfect square**. In general, a perfect square is a quantity that is the square of some other quantity. For example

$$(2x+1)^2 = (2x+1)\times(2x+1) = 2x^2+2x+2x+1 = 4x^2+4x+1$$

so we can say that $4x^2+4x+1$ is a perfect square because it is the square of $2x+1$. Of course the hard part about perfect squares isn't constructing them like this, it is spotting them in the wild. When we complete the square we force a perfect square to exist on the way to solving a quadratic equation. We will start with a very easy example based on the perfect square $(x+1)^2 = x^2+2x+1$.

Example 1.21 *(Completing the square)*

Problem: *Solve* $x^2+2x-1=0$.

Solution:

$x^2+2x-1=0$	*Start with the problem.*
$x^2+2x-1+2=2$	*Add 2 to both sides to make the expression look like the known perfect square, above.*
$x^2+2x+1=2$	*Resolve the arithmetic*
$(x+1)^2=2$	*Factor the quadratic into its perfect square form.*
$x+1=\pm\sqrt{2}$	*Take the square root of both sides, remember that numerical square roots might be positive or negative.*
$x=-1\pm\sqrt{2}$	*Subtract one from both sides. Done.*

Notice that we, again, have two answers: $-1+\sqrt{2}$ *and* $-1-\sqrt{2}$.

Example 1.21 worked out quickly and evenly because x^2+2x-1 is very close to the perfect square $(x+1)^2 = x^2+2x+1$, differing from it only by an additive constant. We can, however, use brute algebra to solve any quadratic that has solutions (or notice that it does not have solutions) with some form of completing the square. We now give the general algorithm for completing the square.

Algorithm 1.2 Completing the square. *Starting with*

$$ax^2+bx+c=0$$

first simplify matters by dividing through by a *to obtain*

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Compute the number $\left(\frac{1}{2}\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$ *and add and subtract it to the left hand side to obtain*

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}=0$$

Notice that $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\left(x-\frac{b}{2a}\right)^2$. *This makes our expression:*

$$\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0$$

Subtracting the terms outside of the square to the right hand side yields

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

At which point simple algebra will solve for x.

Example 1.22 *Completing the square with the algorithm.*

Problem: *Solve* $2x^2-3x-5=0$ *by completing the square.*

Solution: *Follow the algorithm.*

$$\begin{aligned}
2x^2-3x-5 &= 0\\
x^2-\frac{3}{2}x-\frac{5}{2} &= 0\\
x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}-\frac{5}{2} &= 0\\
\left(x-\frac{3}{4}\right)^2-\frac{9}{16}-\frac{5}{2} &= 0\\
\left(x-\frac{3}{4}\right)^2 &= \frac{9}{16}+\frac{5}{2}
\end{aligned}$$

At this point the algorithm is done and we finish with algebra

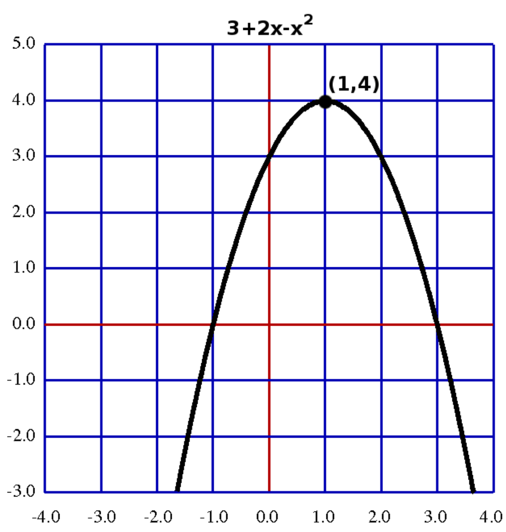
$$\begin{aligned}
\left(x-\frac{3}{4}\right)^2 &= \frac{9}{16}+\frac{5}{2}\times\frac{8}{8}\\
\left(x-\frac{3}{4}\right)^2 &= \frac{9}{16}+\frac{40}{16}\\
\left(x-\frac{3}{4}\right)^2 &= \frac{9+40}{16}\\
\left(x-\frac{3}{4}\right)^2 &= \frac{49}{16}\\
x-\frac{3}{4} &= \pm\sqrt{\frac{49}{16}}\\
x-\frac{3}{4} &= \pm\frac{\sqrt{49}}{\sqrt{16}}\\
x-\frac{3}{4} &= \pm\frac{7}{4}\\
x &= \frac{3}{4}\pm\frac{7}{4}\\
x &= \frac{3\pm 7}{4}
\end{aligned}$$

So in the end we see $x=\frac{-4}{4}=-1$ *or* $x=\frac{10}{4}=\frac{5}{2}$. *In shorthand,* $x=-\frac{5}{2},1$. *This also means that the quadratic factors as*

$$2x^2-3x-5=(2x-5)(x+1)$$

by simply reconstructing the factorization from the roots. The first monomial $(2x-5)$ *is zero when* $x=\frac{5}{2}$ *and the second* $(x+1)$ *is zero when* $x=-1$.

Application: Finding the Vertex of a Parabola



The graph of a quadratic equation is a kind of curve called a **parabola**. If the x^2 term is positive then the parabola opens upward like the ones in Figure 1.4. if it is negative it opens downward like the one shown at the left.

If we complete the square for the quadratic that is graphed at the left we get that

$$-(x-1)^2+4$$

The squared term is zero when $x=1$ and the point on the parabola when $x=1$ is (1,4). This is the point where the parabola turns around. This point is called the **vertex** of the parabola.

Once you've completed the square for any quadratic you can get an expression like $\pm c\cdot(x-u)^2+v$ where c is a constant. The point (u,v) is the vertex of the parabola - the largest value of a parabola opening downward or the smallest value of a parabola opening upward.

Example 1.23 *If the profits in thousands of dollars, based on both cost and potential sales, for manufacturing n advanced military helicopters is given by the formula $P(n) = 360 + 174n - 3n^2$ find the number of helicopters that maximizes the profit.*
Strategy: Since the x^2 term is negative, this quadratic opens downward and so the maximum value is at the vertex. Complete the square:

$$\begin{aligned}
360 + 174n - 3n^2 &= 360 - 3(n^2 - 58n) \\
&= 360 - 3\left(n^2 - 58n + \left(\frac{58}{2}\right)^2 - \left(\frac{58}{2}\right)^2\right) \\
&= 360 - 3\left(n^2 - 58n + 29^2 - 841\right) \\
&= 360 - 3\left(n - 29\right)^2 - 3 \times (-841) \\
&= 360 - 3(n - 29)^2 + 2523 \\
&= 2883 - 3(n - 29)^2
\end{aligned}$$

So we see that

$$P(n) = 2883 - 3(n - 29)^2.$$

Profit is largest when the squared term is zero so this occurs when $n = 29$ and the profit is 2883 thousands or \$2,883,000.

Notice that in Example 1.23 that the numbers were a little large but still worked out evenly. If the number of helicopters at the vertex had come out to something like $n = 31.11657$ then we would need to compare the profits $P(31)$ and $P(32)$. Depending on the exact shape of the profit curve, it might be either one. It is also important to warn you that while completing the square is a method for finding the largest or smallest value of a parabola, when we start using calculus in later chapters we will develop a much more general and robust technique for optimizing.

The Quadratic Formula

At this point we are going to create the quadratic formula by applying the algorithm for completing the square to a general quadratic (one that uses letters for all three constants). We will use a trick of using another name for a complex quantity to keep the expressions simpler than they might otherwise be. Notice that $(x-u)^2 = x^2 - 2u + u^2$. This means we need to force the general form of a quadratic,

$$ax^2 + bx + c$$

into this form in order to complete the square.

Example 1.24 The general form of completing the square.

Problem: *Solve $ax^2 + bx + c = 0$.*

$ax^2 + bx + c = 0$	*Start with the problem.*
$\frac{a}{a}x^2 + \frac{b}{a}x + \frac{c}{a} = \frac{0}{a}$	*Since the form we want starts with x^2, divide through by a to force this to happen.*
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	*Resolve the arithmetic.*
Set $2u = \frac{b}{a}$	*The form we want has $2u$ in front of x, find out what u is in terms of a, b, and c.*
$u = \frac{b}{2a}$	*Divide through by 2.*
$x^2 + 2ux + \frac{c}{a} = 0$	*Substitute the u expression into the original formula.*
$x^2 + 2ux + u^2 - u^2 + \frac{c}{a}$	*Add and subtract u^2 to create the perfect square.*
$x^2 + 2ux + u^2 = u^2 - \frac{c}{a}$	*Subtract all the constants that are not part of the perfect square to the other side.*
$(x + u)^2 = u^2 - \frac{c}{a}$	*Resolve the perfect square.*
$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$	*Get rid of the u by substituting.*
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$	*Start simplifying the right hand side.*
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \times \frac{4a}{4a}$	*Find a common denominator for the fractions on the right.*
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$	*Multiply out the fraction to get the common denominator.*
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$	*Combine the fraction.*
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$	*Take the square root of both sides*
$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}}$	*Subtract the constant on the left from both sides to isolate x.*
$x = -\frac{b}{2a} \frac{\pm\sqrt{b^2-4ac}}{2a}$	*Notice that $4a^2 = (2a)^2$ and simplify the square root. This gives us a common denominator for free!*
$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$	*Combine the fraction. Done.*

Students that recall the quadratic formula will notice that that formula results from completing the square in general.

Fact 1.3 *If $ax^2 + bx + c = 0$ then*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if any such numbers exist.

All three methods, factoring, completing the square, and the quadratic formula can be used to solve any quadratic equation. For different equations, different methods are easier and more effective. There is also the issue that some quadratic equations *do not have solutions.* We will now do some examples of quadratic equations with different numbers of solutions and also explain, in terms of the graphs of the equations, why a given quadratic has the number of solutions it does.

Fact 1.4 Finding the number of solutions of a quadratic
The quadratic formula involves plus-or-minus a square root. The way that square root works out predicts the number of solutions to the equations $ax^2 + bx + c = 0$ by the following rules:

If $b^2 - 4ac > 0$ then there are two solutions,

if $b^2 - 4ac = 0$ then there is one solution,

and if $b^2 - 4ac < 0$ then there are no solutions.

The quantity $b^2 - 4ac$ is called the discriminant *of the quadratic.*

When the rule for the number of solutions of a quadratic says there are no solutions, this means no solutions that take on real number values. There *are* solutions but they are not real numbers. You may have encountered them in your previous experience in mathematics, they are called *complex numbers.* We don't work with complex numbers in these notes, but many online resources are available if you are curious.

Exercises

Exercise 1.21 *For each of the following lines, state the slope and intercept.*

a) $y = 2x + 5$. **b)** $y = -3x + 1$. **c)** $2x - 4y = 5$. **d)** $3(x + y) + 2x = 4$. **e)** $(y - 5) = 3(x - 1)$.

f) The line through (1,1) and (3,5).

g) A line parallel to $y = -2x + 5$ through the point (-1,3).

h) The line intersecting $y = (x - 5)/2$ at right angles at the point (3,-1).

i) The line of slope 2 through the point (a,b).

j) A line intersecting $y = 2abx + 1$ at right angles at the point (1,2ab+1).

Exercise 1.22 *Find both the point-slope and slope-intercept forms of lines through the following pairs of points.*

a) (1,1) and (2,5) **b)** (1,1) and (5,2) **c)** (-1,2) and (2,-1) **d)** (17,122) and (35,-18)

e) (a,b) and (3,2) **f)** (x, x^2) and $(x + 2, (x + 2)^2)$

Exercise 1.23 *Find the intersections of the following sets of lines. Hint: at least one pair of lines does* not *intersect.*

a) $y = 2x + 1$ and $y = -x + 10$. **b)** $y = 2x + 1$ and $y = 3x - 5$. **c)** $y = 2x + 1$ and $2y - 4x = 3$.

d) $2x + 3y = 1$ and $3x - 5y = 7$. **e)** $x + y + 1 = 0$ and $x - y - 1 = 0$. **f)** $y = ax + 2$ and $y = -2x + 5$.

Exercise 1.24 *Carefully graph the following lines.*

a) $y = 3x - 2$. **b)** $y = 7x + 1$. **c)** $y = -2x + 3$. **d)** $y = -x/3 + 2/3$. **e)** $y = 6 - x$. **f)** $2y + 5x = 1$.

Exercise 1.25 *Factor the following quadratic equations.*

a) $x^2 + 4x + 3$. **b)** $x^2 - 8x + 15$. **c)** $x^2 + 2x - 8$. **d)** $x^2 - 6x + 9$. **e)** $9x^2 - 6x + 1$. **f)** $x^2 - 25$. **g)** $x^2 - 12$. **h)** $2x^2 + 5x - 3$. **i)** $6x^2 + 13x + 6$. **j)** $x^2 + \frac{1}{2}x + \frac{1}{4}$.

Exercise 1.26 *Complete the square to place each of the following quadratics in the form $\pm c \cdot (x - u)^2 + v$ for some constants c, u and v. Also state if each of the parabolas opens up or down.*

a) $x^2 - 8x + 60$. **b)** $x^2 - 15x + 12$. **c)** $22 + 6x - x^2$

d) $126 + 20x - 2x^2$ **e)** $(x + 4)(x - 24)$ **f)** $4x^2 - 12x + 14$

Exercise 1.27 *For each of the following, use the quadratic equation to find all the solutions or demonstrate with the discriminant that there are none.*

a) $x^2 + x + 1 = 0$. **b)** $x^2 - x - 1 = 0$. **c)** $x^2 + 2x + 1 = 0$ **d)** $2x^2 + 5x - 7$

e) $2x^2 + 5x + 7$ **f)** $7 - 2x - x^2 = 0$ **g)** $x^2 = 2$

h) $-2x^2 + 5x + 6 = 0$ **i)** $(x+1)(x-6) = 4$ **j)** $17.1x^2 - 122.8x - 76.3 = 0$

Exercise 1.28 *Carefully graph the following quadratics.*

a) $y = x^2$. **b)** $y = x^2 - 4x + 4$. **c)** $y = (x-3)^2 - 5$. **d)** $y = x^2 + x - 1$. **e)** $y = x^2/4 + x/2 - 1$. **f)** $y = (x+2)(x-2) + 1$.

Exercise 1.29 *Graph the set of all points (a,b) so that the line through the points (a,b) and (3,2) has a slope of 2.*

Exercise 1.30 *Graph the set of all points (u,v) so that the line through the points (a,b) and (-1,2) has a slope of 3/2.*

Exercise 1.31 *Find two different point-slope forms for the line* $y = 2x - 5$.

Exercise 1.32 *Find two different point-slope forms for the line* $2y - 6x + 2y = 22$.

Exercise 1.33 *Prove or disprove that the following points are the vertices of a right triangle: (1,3), (-6/5,-7/5), (2,-1).*

Exercise 1.34 *Prove or disprove that the following points are the vertices of a right triangle: (1,3), (-3,-1), (2,-1).*

Exercise 1.35 *Find four lines that enclose a square of area 9.*

Exercise 1.36 *Find four lines, one of which has slope 2, so that the four lines enclose a square with area 4.*

Exercise 1.37 *Suppose that the supply curve for a particular brand of MP3 player is* $p = q/25$ *while the demand curve is* $p = 1000 - q/30$. *Find the quantity and price that balance supply and demand.*

Exercise 1.38 *Suppose that the supply curve for light electric scooters is* $p = 20 + q/1250$ *while the demand curve is* $p = 400 - q/1000$. *Find the quantity and price that balance supply and demand.*

Exercise 1.39 *Find a quadratic* $y = -x^2 + bx + c$ *that opens downward and has roots at* $x = -2$ *and* $x = 7$. *Find its maximum value by completing the square.*

Exercise 1.40 *Find a quadratic* $y = -x^2 + bx + c$ *that opens downward and has roots at* $x = 1$ *and* $x = 15$. *Find its maximum value by completing the square.*

Exercise 1.41 *If the profit for making* n *video game consoles is given by*

$$P(n) = 25000 + 850n - 10n^2$$

then complete the square to find the vertex of the parabola and find the number of consoles that maximizes profit.

Exercise 1.42 *If the profit for printing a particular poster is given by the formula*

$$P(n) = 4.2 + 3.8n - 0.014n^2$$

then complete the square to find the vertex of the parabola and find the number of posters that maximizes profit.

1.3 Exponents, Exponentials, and Logarithms

In this section we review the rules for exponents, extending the notion to exponents that are not whole numbers and then go on to work with exponential and logarithmic functions.

1.3.1 Exponents: negative and fractional

You are already familiar with the idea of raising a variable to a power. The expression x^2 means multiply x by itself; x^3 means multiply three copies of x. It turns out that both negative and fractional exponents, like x^{-2} and $x^{3/4}$ also have meanings. We will build them up now, one step at a time.

The notation $a = \sqrt[n]{c}$ means that a is a number so that $a^n = c$. The name for this number is *the nth root of c*. Fractional exponents arise from roots via the following convention:

$$\sqrt[n]{c} = c^{\frac{1}{n}}$$

This convention extends to fractions that are not just reciprocals of whole numbers by taking powers of roots or roots of powers:

$$\sqrt[n]{c^m} = \left(\sqrt[n]{c}\right)^m = c^{m/n}$$

Example 1.25 Problem: *compute* $8^{2/3}$.

Using the convention above,

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

Notice that this example works out particularly neatly, with a whole number answer.

Fact 1.5 *Reciprocals are negative first exponents.*

So for example, $2^{-1} = \frac{1}{2}$ and $x^{-2} = \frac{1}{x^2}$. Here are some more rules of algebra, the ones that apply to roots and exponents.

1. $a = a^1$ (mostly used when simplifying).
2. Negative numbers do not have even roots; they do have odd roots.
3. If $a = b$ then $a^c = b^c$ unless there is a problem with a and b being negative.
4. $\sqrt[b]{a} = a^{1/b}$.
5. $a^b \cdot a^c = a^{b+c}$.
6. $\frac{a^b}{a^c} = a^{b-c}$.
7. $\left(a^b\right)^c = a^{bc}$.
8. $\frac{1}{a^b} = a^{-b}$.

While all the rules are phrased in terms of exponents, the fact that roots are fractional exponents mean they apply to roots as well. Rule 4 provides the connection.

Notice, also, that odd roots and powers preserve negative signs. Even roots, like the square root, require a $\pm$ - they can produce a positive or negative answer. This fact goes hand-in-hand with the fact that even powers forget minus signs.

Example 1.26 Using fractional exponents
Problem: *Solve* $x^{3/4} = 2$.

The reciprocal of 3/4 is 4/3 so a good way to get rid of the fractional power is to take the 4/3rds power of each side and then use normal algebra:

$$
\begin{aligned}
x^{3/4} &= 2 \\
\left(x^{3/4}\right)^{4/3} &= 2^{4/3} \\
x^{\frac{3}{4}\times\frac{4}{3}} &= \sqrt[3]{2^4} \\
x^1 &= \sqrt[3]{16} \\
x &= \sqrt[3]{16} \cong 2.52
\end{aligned}
$$

Example 1.27 Solving a strange type of quadratic with fractional exponents.
Problem: *Solve:* $x^{4/3} - 3x^{2/3} + 2 = 0$

The tricky step in this problem is to notice that $x^{4/3} = \left(x^{2/3}\right)^2$. *Once you notice this, rewrite the problem as a quadratic-like expression and solve it, in this case by factoring.*

$$
\begin{gathered}
x^{4/3} = \left(x^{2/3}\right)^2 \\
\left(x^{2/3}\right)^2 - 3\left(x^{2/3}\right) + 2 = 0 \\
\left(x^{2/3} - 1\right)\left(x^{2/3} - 2\right) = 0 \\
x^{2/3} = 1 \text{ or } x^{2/3} = 2 \\
x = 1^{3/2} \text{ or } x = 2^{3/2} \\
x = 1 \text{ or } x = \sqrt{2^3} \\
x = 1 \text{ or } x = \sqrt{8}
\end{gathered}
$$

and we have the answer.

Example 1.28 Clearing a root from the denominator

Problem: *simplify the fraction* $\frac{\sqrt{x-1}+2}{\sqrt{x-1}-1}$. *In particular, eliminate the square-root in the denominator.*

Strategy: use the fact that $(a-b)(a+b) = a^2 - b^2$ *to clear the denominator.*

$$
\begin{aligned}
\frac{\sqrt{x-1}+2}{\sqrt{x-1}-1} &= \frac{\sqrt{x-1}+2}{\sqrt{x-1}-1} \times \frac{\sqrt{x-1}+1}{\sqrt{x-1}+1} \\
&= \frac{x-1+3\sqrt{x-1}+2}{(x-1)-1} \\
&= \frac{x+1+3\sqrt{x-1}}{x-2}
\end{aligned}
$$

And the square-root is removed from the denominator.

So far all the functions we've dealt with can be created by doing arithmetic on variables and constants. At this points we introduce *functions*. A function is a a mathematical widget that accepts a number and returns another number. An example with which you are already familiar is the square root. Square roots cannot accept any number - they only return values if the number you are putting into the square root is not negative.

We are introducing two new types of functions in this section: exponentials and logarithms. These two are opposite to one another, like the square and square-root and they have their own collection of algebraic rules. We start by defining them.

Definition 1.1 *An* **exponential function** *is any function of the form*

$$y = c^x$$

where c is a positive constant. Examples of exponential function include $y = 2^x$ or $y = \left(\frac{1}{4}\right)^x$.

The partner to the exponential function is the logarithm function. This is a little trickier to define and we will start not with the function but with individual logarithms.

Definition 1.2 *Suppose that a, b, and c are constants and that $b > 0$. Then $log_b(a) = c$ if and only if $b^c = a$. When $log_b(a) = c$ we say "The log base b of a equals c."*

Example 1.29 Examples of logarithms

Since $2^3 = 8$ we can say that $log_2(8) = 3$

Since $10^2 = 100$ we can say that $log_{10}(100) = 2$.

If $a = \sqrt{3}$ then the fact $a = 3^{1/2}$ means $log_3(a) = \frac{1}{2}$.

Since $\frac{1}{1000} = 10^{-3}$ we can say that $log_{10}(\frac{1}{1000}) = -3$.

Most logs are not whole numbers: $log_{10}(2) \cong 0.30103$

With those examples of logarithms in hand we can define logarithmic functions.

Definition 1.3 Logarithmic functions *are written $y = log_b(x)$. Because a power of a positive number b must be positive, logarithm functions only exist when $x > 0$.*

If we have the functions $y = c^x$ or $y = log_b(x)$ then we call c and b the **base** of the exponential or logarithmic functions, respectively. We require that the base of an exponential or logarithmic function be positive. Functions with a base of one are exceptional; the exponential function is a constant function equal to one while the log function is the vertical line $x = 1$. Because of this the base of one is not much use and seldom seen. There are two logarithmic functions that, by convention, do not require us to state the base.

Definition 1.4 The usual logarithm functions.
We make the following pair of notational conventions:

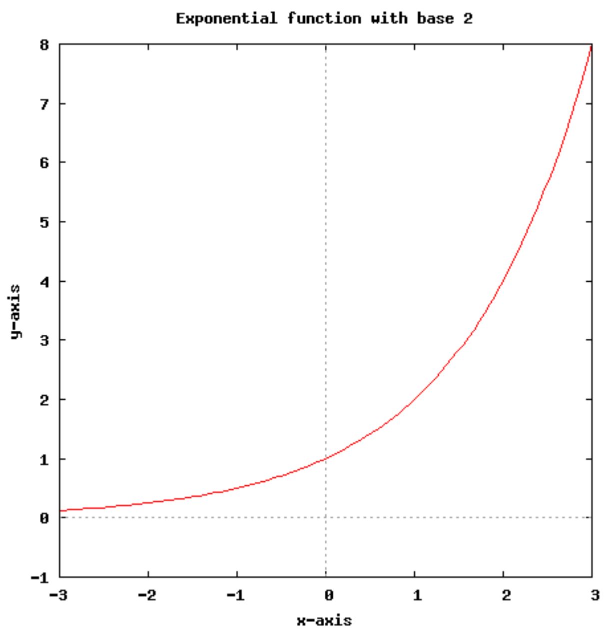
$log(x)$ means $log_{10}(x)$, and

$ln(x)$ means $log_e(x)$

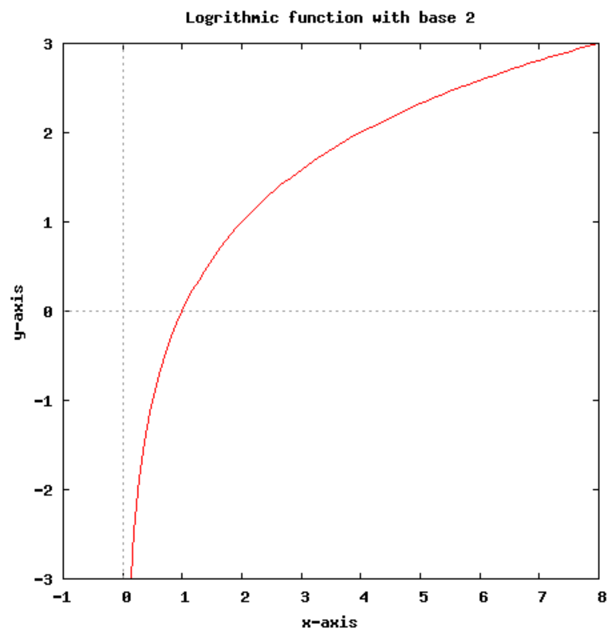
The second function is called the natural log *and the number $e \cong 2.7182818$. Getting an exact value for e requires infinite series and is covered in the next chapter.*

Any logarithm function other than the two above require an explicit statement of their base, with the exception of some computer science classes where $log(x)$ is confusingly used as a shorthand name for $log_2(x)$. The number e is a special type of constant, like π, that arises in a natural fashion out of mathematics itself. Both e and π have infinite decimal expansions that do not ever settle down to a repeating pattern.

Fact 1.6 Graphs of the exponential and logarithmic functions *The functions $y = 2^x$ and $y = log_2(x)$ are graphed below. The shape of these functions are common to all exponential and logarithmic functions which have a base greater than one.*



$y=2^x$



$y=log_2(x)$

Geometric properties of exponential functions.

1. *The exponential function exists for all values of x.*

2. *It can take on any positive value but only takes on positive values.*

3. *The function increases from left to right if the base is bigger than 1 and decreases if the base is smaller than one.*

4. *As x grows, an exponential function with a base larger than one grows rapidly toward infinity.*

5. *As x becomes larger in the negative direction, an exponential function with a base larger than one approaches arbitrarily close to zero.*

Geometric properties of logarithmic functions.

1. *The logarithmic function exists only for positive values of x.*

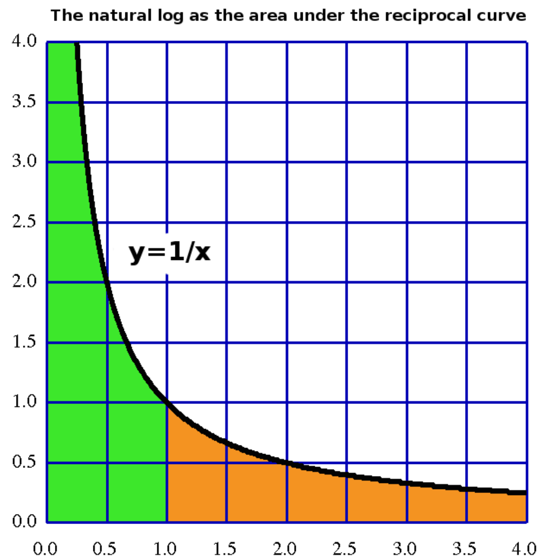
2. *It can take on any value of y.*

3. *The function increases from left to right if the base is bigger than 1 and decreases if the base is smaller than one.*

4. *As x grows, a logarithmic function with a base exceeding one grows slowly toward infinity.*

5. *As x approaches zero, a logarithmic function with a base exceeding one rapidly toward negative infinity.*

Why base e?



Logarithm functions were used, before we had machine computation, to make multiplication and exponentiation easier. Where, though, did the function come from? Why is e considered the natural base? Examine the graph at the left. The area under the curve $y = 1/x$ and the x-axis, between 1 and c is equal to $ln(c)$. This means that the log function that arises "naturally" from the rest of mathematics is the log base e.

The orange areas, to the right of $x = 1$, correspond to the natural log of numbers bigger than one, which are positive. The green areas, to the left of $x = 1$, correspond to the natural log of numbers smaller than one, which are negative. Areas are, of course, always positive, but the negative of the area from $c < 1$ to $x = 1$ is the natural log of such c.

One technique that was used to estimate logarithms is to make a graph like the one above on a piece of heavy paper of known area. The paper was weighed and then the area corresponding to the log was then cut out. The ratio of the weight of the cut-out area to the whole sheet yields an estimate of the corresponding log.

The exponential and logarithmic functions are presented together because they are intimately linked. Much as the square and square root have the opposite effect, exponentials and logarithms have opposite effects: each can be used to undo the other. The analogy even holds to the following extent: you can take the square (exponential) of anything but only some numbers have square roots (logarithms). These properties are stated in mathematical form in the following list of properties.

Exponential and logarithmic functions have the following algebraic properties. Many of these may be used to simplify expressions. A couple of the rules are repeated from the section on roots and exponents - this is done to make this list complete.

1. If $a = b$ then $c^a = c^b$ for any $c > 0$. (exponentiate with both sides).
2. If $a = c$ then $log_b(a) = log_b(c)$ whenever $a > 0$. (take the log of both sides).
3. $log_b(a^c) = c \times log_b(a)$
4. $log_b(a \times c) = log_b(a) + log_b(c)$
5. $log_b(a/c) = log_b(a) - log_b(c)$
6. $b^a \times b^c = b^{a+c}$
7. $(b^a)^c = b^{a \times c}$
8. $b^{log_b(a)} = a$ for $b > 0$. (for getting rid of logs).
9. $log_b(b^a) = a$ for $b > 0$. (for getting rid of exponentials).
10. $\frac{log_b(a)}{log_b(c)} = log_c(a)$ (base conversion law).

Notice that the last of the properties above tells you how to compute logs for any base even if you only have a calculator that does log base 10 or base e. We are now ready to practice some problem solving with logs and exponents.

Example 1.30 Problem: *Compute $log_3(7)$.*

Using rule 10 above,

$$log_3(7) = \frac{log(7)}{log(3)} \cong \frac{0.84509804}{0.47712125} \cong 1.7712437$$

Note that we are using the log base 10 to solve the problem. The natural log would have worked as well:

$$log_3(7) = \frac{ln(7)}{ln(3)} \cong \frac{1.9459101}{1.0986123} \cong 1.7712437$$

Only the intermediate values are different.

Example 1.31 Problem: *Solve $3 = 2^x$ for x.*

Strategy: take the log base two of both sides.

$$\begin{aligned} 3 &= 2^x \\ log_2(3) &= log_2\left(2^x\right) \\ log_2(3) &= x \\ x &= log_2(3) \\ x &= \frac{ln(3)}{ln(2)} \cong 1.5849625 \end{aligned}$$

Example 1.32 Problem: *Solve $ln(x+3) = 2$ for x.*

Strategy: first remove the log with an exponential, then resolve the expression with standard algebra. Remember that $ln(x)$ is $log_e(x)$.

$$\begin{aligned} ln(x+3) &= 2 \\ e^{ln(x+3)} &= e^2 \\ x+3 &= e^2 \\ x &= e^2 - 3 \cong 4.3890561 \end{aligned}$$

Example 1.33 Problem: *Solve $4^x - 3 \cdot 2^x + 2 = 0$ for x.*

Strategy: Notice that $(2^x)^2 = 4^x$ *and treat the problem as a factorable quadratic.*

$$\begin{aligned}
4^x - 3 \cdot 2^x + 2 &= 0 \\
(2^x)^2 - 3\,(2^x) + 2 &= 0 \\
(2^x - 1)\,(2^x - 2) &= 0 \\
&\textit{SO} \\
2^x - 1 &= 0 \\
2^x &= 1 \\
log_2(2^x) &= log_2(1) \\
x &= 0 \\
&\textit{OR} \\
2^x - 2 &= 0 \\
2^x &= 2 \\
log_2(2^x) &= log_2(2) \\
x &= 1
\end{aligned}$$

And as can happen with quadratic or quadratic-like equations we get two answers: $x = 0, 1$.

The examples in this section give you a starter set of methods for using logs and exponentials in algebra. There are some pitfalls. If you are working a problem and you need to take the log of a negative number then, if you didn't make mistakes, the correct answer is that the answer is undefined. If either of the roots of the odd quadratic in Example 1.33 had been negative that would have meant that there was no answer associated with that root.

In a similar fashion, if an exponential function that needs to produce a negative number to solve a problem, then the problem may not have a solution. The way to write such a non-existent answer varies but two common notations for these situations where something impossible happens are **undefined** or **does not exist**.

Application: Compound Interest

If an account pays 5% simple interest per year, that means that at the end of each year, the bank adds money equal to 5% of the current total to your account. This, in effect, multiplies the account by 1.05 each year. This means that the amount of money in the account, is the original amount was D dollars, the amount of money at the end of n years is

$$D \times (1.05)^n$$

in other words the original amount multiplied by 1.05 n times.
Problem: how many years to double, or more than double, the account?

$$\begin{aligned} 2D &= D \cdot (1.05)^n \\ 2 &= 1.05^n \\ log_{1.05}(2) &= log_{1.05}(1.05)^n \\ \frac{log(2)}{log 1.05} &= n \\ n &= \frac{log(2)}{log(1.05)} \cong 14.206699 \end{aligned}$$

And so the amount of money passes double the original amount at the end of year fifteen.

Exercises

Exercise 1.43 *Simplify each of the following expressions as much as possible.*

a) $\left(2^2\right)^3$. **b)** $\left(x^3\right)^2$. **c)** $x^2 \times x^{3/4} \div x^{5/2}$. **d)** $(2x)^2 \div (4x)^{1/2}$. **e)** $\frac{2+2^{-1}}{2^{-2}+2^{-3}}$. **f)** $\frac{x^{1/2}y^{3/4}}{x^2y^{4/3}}$.

Exercise 1.44 *Solve the following equations.*

a) $x^6 - 5x^3 + 6 = 0$. **b)** $x - 6\sqrt{x} + 8 = 0$. **c)** $e^{2x} - 5e^x + 4 = 0$. **d)** $e^{2x} - 4 = 0$.

e) $4x^{3/4} - 4x^{3/8} + 1 = 0$. **f)** $xe^x - x - e^x + 1 = 0$ (hint: factor). **g)** $x^{-2} + x^{-1} - 2 = 0$.

h) $x^{-2} + x^{-1} - 1 = 0$. **i)** $9^x - 4 = 0$. **j)** $4^x - 5 \cdot 2^x + 6 = 0$.

Exercise 1.45 *Solve the following equations using logs and exponentials as needed.*

a) $log(x + 1) = 4$. **b)** $e^{x+1} = 4$. **c)** $log_2(x - 2) = 3$. **d)** $1.05^x = 3.5$. **e)** $log_x(12) = 5$. **f)** $3^x = 14$. **g)** $log_{1/4}(x) = 2$. **h)** $log\left(x^2 + 4x + 4\right) = 1$. **i)** $Ln\left(\frac{x-1}{x+1}\right) = 1$. **j)** $2 = \frac{e^x+1}{e^x-1}$.

Exercise 1.46 *Remove the radical from the denominator of* $\frac{\sqrt{x}-1}{\sqrt{x}+1}$.

Exercise 1.47 *Remove the radical from the denominator of* $\frac{\sqrt{2x+5}-2}{\sqrt{2x+2}-3}$.

Exercise 1.48 *Remove the radical from the denominator of* $\frac{2x}{1-\sqrt{2x}}$.

Exercise 1.49 *Remove the radical from the denominator of* $\frac{x^2+1}{7-\sqrt{x}}$.

Exercise 1.50 *Carefully graph the following functions using an appropriate range and domain.*

a) $y = log_3(x)$. **b)** $y = log_{1/2}(x)$. **c)** $y = ln(3x)$. **d)** $y = 1.1^x$. **e)** $y = 2^{x+4}$. **f)** $y = e^{x \times ln(3)}$.

Exercise 1.51 *If* $log(a) = 1.5$ *and* $log(b) = 2.25$ *then what is the logarithm of* $log\left(\frac{a^3}{b^2}\right)$*?*

Exercise 1.52 *If* $log(u) = 0.5$ *and* $log(v) = -1$ *then what is the logarithm of* $log\left(\frac{u^2}{v^3}\right)$*?*

Exercise 1.53 *What is the smallest whole number* n *so that* $1.15^n > 3$*?*

Exercise 1.54 *What is the smallest whole number* n *so that* $1.03^n > 2.5$*?*

Exercise 1.55 *If $1000.00 is invested at d% interest compounded yearly for 10 years and the final balance is $1218.99 then what was the rate of interest?*

Exercise 1.56 *If $400.00 is invested at d% interest compounded yearly for 8 years and the final balance is $637.54 then what was the rate of interest?*

Exercise 1.57 *If $1000.00 is invested at 5% interest for 10 years how much of a difference does it make if the interest is compounded yearly or monthly? In the latter case* $\frac{5}{12}$*% interest is paid each month.*

Exercise 1.58 *If $400.00 is invested at 6% interest for 12 years how much of a difference does it make if the interest is compounded yearly or monthly? In the latter case* $\frac{1}{2}$*% interest is paid each month.*

Exercise 1.59 *Suppose that a person borrows $1000.00. If they pay the same amount every month and also pay 0.5% monthly interest on the remaining balance what payment will clear the debt in one year? Hint: you need a high-lo game to solve this problem.*

1.4 Moving Functions Around

This section deals with moving and distorting the graphs of functions. The following rules cover horizontal and vertical shifts.

1. The graph of $f(x - c)$ is the graph of $f(x)$ shifted c units to the right.
2. The graph of $f(x + c)$ is the graph of $f(x)$ shifted c units to the left.
3. The graph of $f(x) + c$ is the graph of $f(x)$ shifted c units upward.
4. The graph of $f(x) - c$ is the graph of $f(x)$ shifted c units downward.

Figure 1.5 shows the result of applying horizontal and vertical shifts to a simple parabola. This type of action is called a *translation* of the graph of the function. It amounts to a rigid move of the graph that does not change its shape at all.

There are two simple ways to bend the shape of a graph.

1. The graph of $c \times f(x)$ is the graph of $f(x)$ with the y-axis stretched (multiplied by) a factor of c. If $c < 1$ then the "stretch" is really a compression because everything gets closer to the x axis. If $c < 0$ then the graph is flipped around the x-axis.
2. The graph of $f(c \times x)$ is the graph of $f(x)$ with the x-axis stretched (multiplied by) a factor of $\frac{1}{c}$. If $c > 1$ then the "stretch" is really a compression because everything gets closer to the y axis. If $c < 0$ then the graph is flipped around the y-axis.

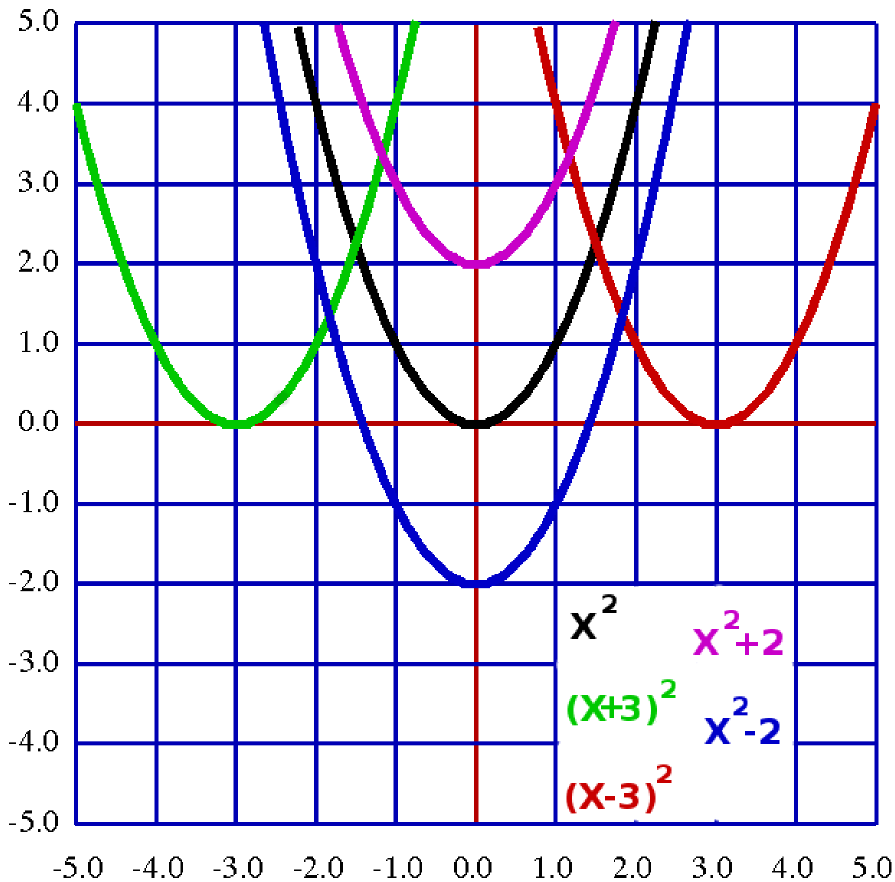


Figure 1.5: The result of shifting the graph of $f(x) = x^2$ three units to the left and right and two units up and down.

Figure 1.6 shows the result of bending the function $f(x) = \left(x^3 - 4x\right)/2$ in various ways. It turns out that translating and bending functions is closely related to completing the square.

Example 1.34 Problem: *Complete the square to show how $y = x^2 - 4x + 2$ is a bent translation of $f(x) = x^2$.*

$$\begin{aligned} y &= x^2 - 4x + 4 - 4 + 2 \\ y &= (x-2)^2 - 2 \end{aligned}$$

So we see that the graph of $y = x^2 - 4x + 2$ is the graph of $f(x) = x^2$ shifted two units to the right and two units down.

Notice that, in the graphs shown in Figure 1.6 that all five of the graphs cross the x-axis at zero and all but $f(2x)$ share the axis crossings at $x = \pm 2$. This is true simply because zero times anything is still zero. This nice fact does not hold when we translate graphs. If you translate the graph of a function then its roots, the places where it is zero, can move (and usually do). When you modify the x-axis as in $f(2x)$ the roots can also move.

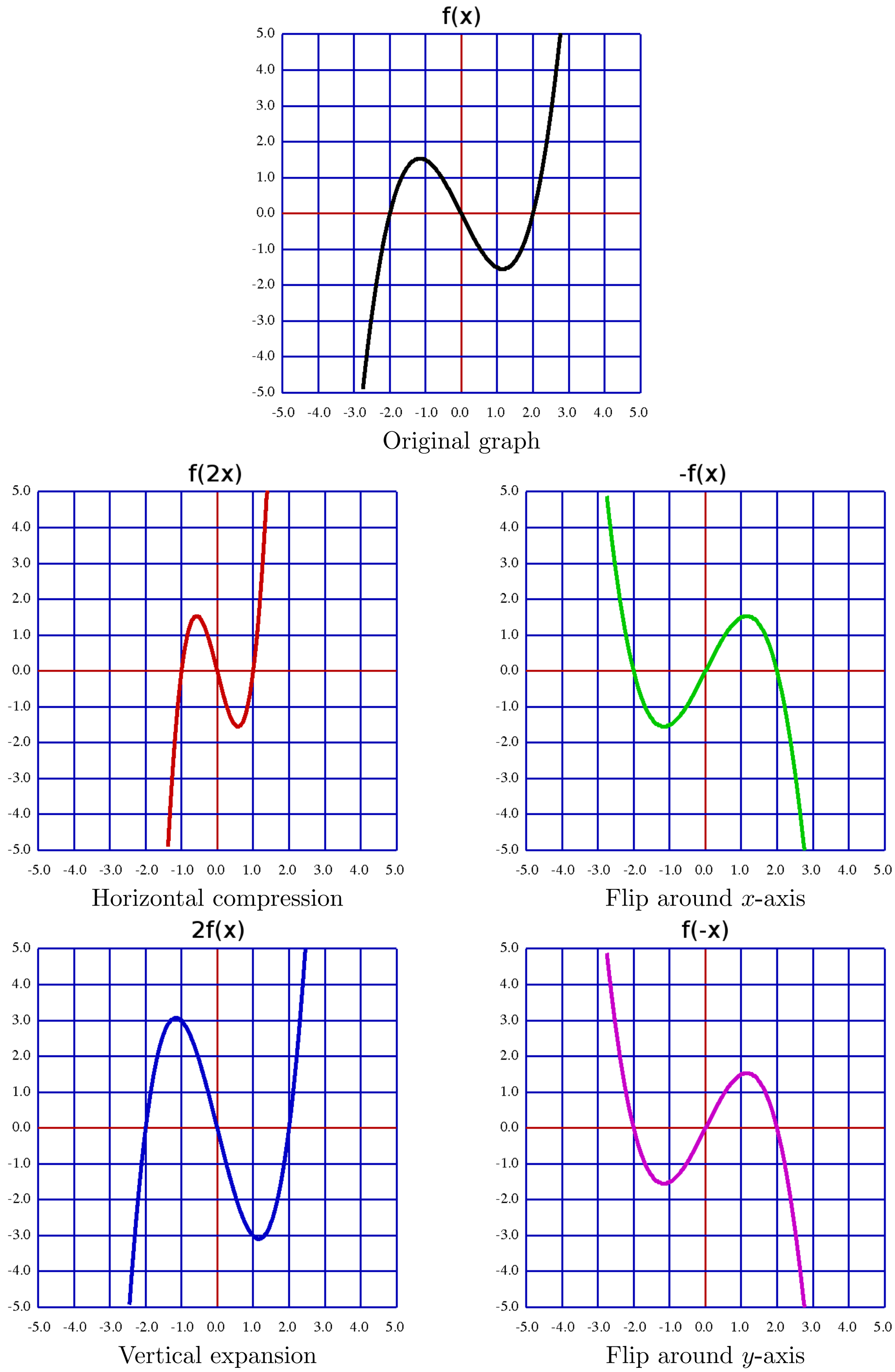


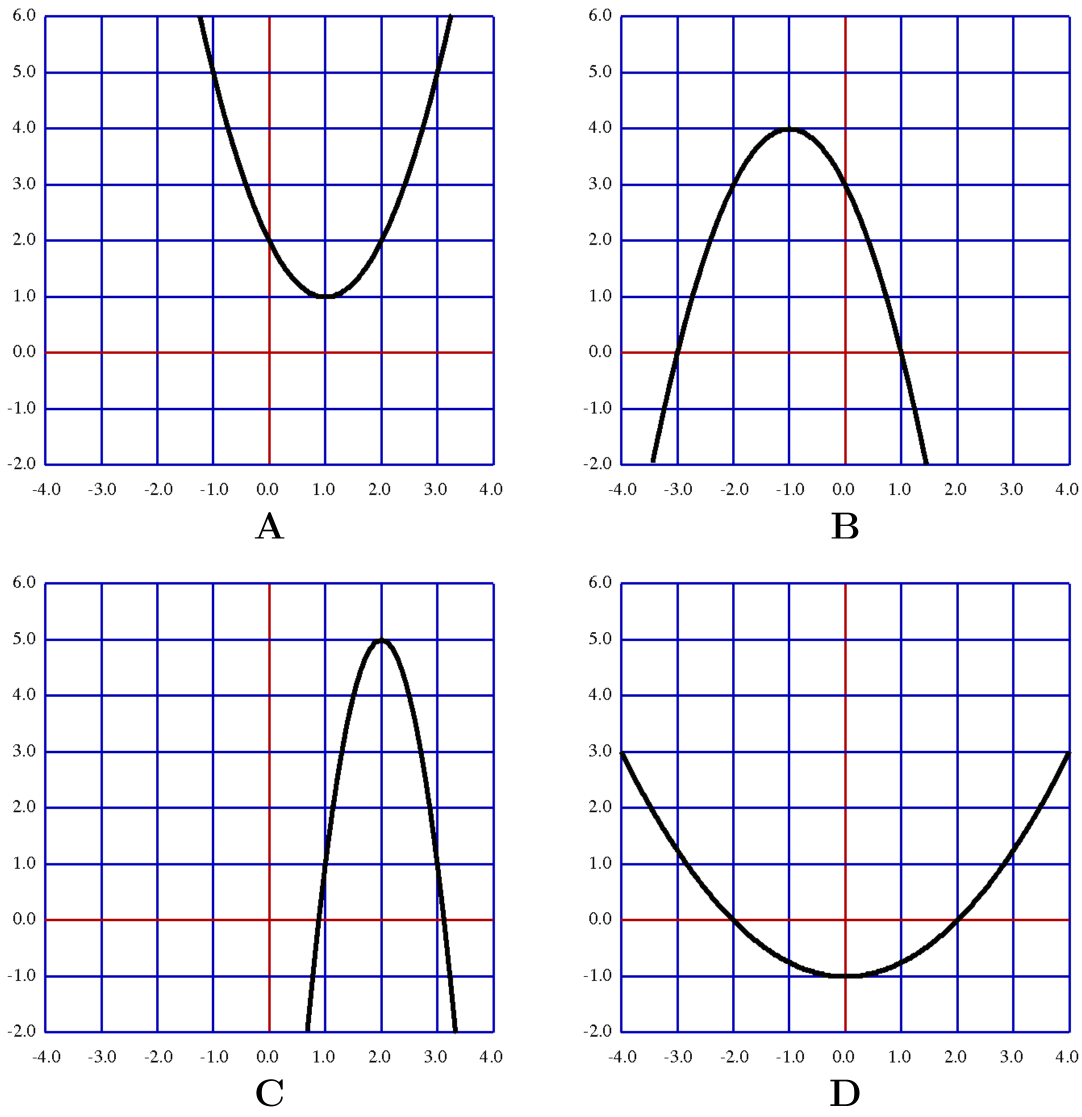
Figure 1.6: The result of applying distortion factors of 2 or -1 to the cubic function $f(x) = \frac{1}{2}\left(x^3 - 4x\right)$. Notice that in this case $f(-x) = -f(x)$. This is not always the case, but in this example it shows that apparently different modifications of a function may have the same effect.

Exercises

Exercise 1.60 *If* $f(x) = x^3 - x$ *carefully graph* $f(x)$, $f(x-2)$, $f(2x)$ *and* $f(x) + 2$.

Exercise 1.61 *If* $f(x) = \sqrt{x^2+1}$ *carefully graph* $f(x)$, $f(x-3)$, $f(2x)$ *and* $f(x) + 1$.

Exercise 1.62



Each of the graphs above is the result of translating, and in some cases bending, $f(x) = x^2$ *so that the function is* $f(x) = a(x-b)^2 + c$. *For each of the above find a, b, and c.*

Exercise 1.63 *Translate the function* $f(x) = x^2 + 2x$ *by finding a and b to place the vertex of* $f(x-a) + b$ *at (1,1).*

Exercise 1.64 *Translate the function* $f(x) = x^2 - 4x$ *by finding a and b to place the vertex of* $f(x-a) + b$ *at (2,-1).*

Exercise 1.65 *Use translation and bending to make the function $f(x) = 2^x$ go through the points (0,2) and (1,5).*

Exercise 1.66 *Use translation and bending to make the function $f(x) = 3^x$ go through the points (0,-1) and (2,6).*

1.5 Methods of Solving Equations

In this section we will put a little polish on our techniques for solving equations and learn a last-resort method that works on almost anything but is a lot of work to apply. We will start with an example.

Example 1.35 Problem: *Solve $x^3 - 4x = 0$.*

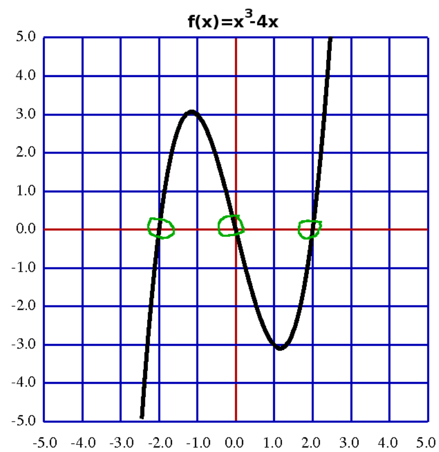
Strategy: factor the expression. Since $x^3 - 4x = x(x^2 - 2)$ it is easy to see that

$$x^4 - 4x = x(x-2)(x+2)$$

We then apply the rule that if numbers multiplied together are zero, one of the numbers must be zero, and get three answers:

$x = 0$,
$x - 2 = 0$ *so* $x = 2$, *and*
$x + 2 = 0$ *so* $x = -2$

If we were to graph the function $f(x) = x^3 - 4x$:



then we see that the solutions $x = 0, \pm 2$ are all places where the graph crosses the x-axis. These have been circled in green.

Remember that a *root* is a number that makes a function zero and a *factor* of an expression is another expression that divides it evenly.

Fact 1.7 The root-factor theorem *If $f(c) = 0$ then $(x - c)$ divides $f(x)$ evenly.*

Example 1.36 Problem: *Factor $f(x) = x^3 - 6x^2 + 11x - 6$.*
Strategy: Plug in small numbers looking for a root and apply the root-factor theorem.

$$f(0) = -6,\ f(1) = 1 - 6 + 11 - 6 = 0 \text{ Aha!}$$

The root factor theorem tells us $(x-1)$ is a factor, so now we divide:

$$\begin{array}{r|l} & x^2 \quad -5x+6 \\ \hline x-1) & x^3-6x^2+11x-6 \\ & -x^3 \quad +x^2 \\ \hline & \quad -5x^2+11x \\ & \quad 5x^2 \quad -5x \\ \hline & \qquad\qquad 6x-6 \\ & \qquad\qquad -6x+6 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

The fact the division comes out evenly tells us that

$$x^3-6x^2+11x-6=(x-1)(x^2-5x+6)$$

If we factor the quadratic this gives us

$$x^3-6x^2+11x-6=(x-1)(x-2)(x-3)$$

and so if we wanted to solve $x^3-6x^2+11x-6=0$ the answers would be $x=1,2,3$.

In this section we are working with functions that involve x^3 where before we were working mostly with lines and quadratics. This is mostly because the root-factor theorem only starts to be interesting when you're dealing with things bigger than quadratic equations. We now make some definitions that give us vocabulary to talk about this sort of expression in general.

Definition 1.5 *A* **polynomial** *is a sum of constant multiples of non-negative whole-number powers of a variable. Examples include $2x+1$, x^2+x+1, x^3+2x^2-4x-7, and x^7+3x-1. The* **degree** *of a polynomial is the largest exponent of the variable so the degrees of the examples are 1, 2, 3 and 7. A* **polynomial function** *is a polynomial used as the rule of a function as in $y=x^2+x+1$.*

The first example using the root factor theorem came out nice and even. We now do a similar example that does not come out quite so nicely.

Example 1.37 Problem: *Solve $x^3+3x^2+x-2=0$.*
Strategy: Plug in small numbers looking for a root and apply the root-factor theorem.

$$f(0)=-2,\ f(1)=3,\ f(-1)=-1,\ f(2)=20,\ f(-2)=0 \ \textit{Aha!}$$

Since $x=-2$ is a root the root-factor theorem tells us that $(x+2)$ is a factor, so we divide:

$$\begin{array}{r|l} & x^2 \quad +x-1 \\ \hline x+2) & x^3+3x^2 \quad +x-2 \\ & -x^3-2x^2 \\ \hline & \qquad x^2 \quad +x \\ & \quad -x^2-2x \\ \hline & \qquad\qquad -x-2 \\ & \qquad\qquad\quad x+2 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

We get the one obvious answer $x=2$ but we also have to solve $x^2+x-1=0$. Since this does not factor evenly, we use the quadratic formula and get two more answers:

$$x = \frac{-1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2} \cong 0.61803399 \ or \ -1.618034$$

If it has been a while since you did long division on polynomials, you should probably work some of the practice problems in the exercises. We conclude this section with some handy pre-factored expressions. If you can recognize these while solving a problem they can serve as a shortcut. They usually are useful when working with polynomials but may come up in other contexts.

Fact 1.8 Polynomial general forms

1. $a^2 - b^2 = (a-b)(a+b)$
2. $(a+b)^2 = a^2 + 2ab + b^2$
3. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
4. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Example 1.38 Examples of general forms
Here are some examples of applying the general forms.

- $x^2 - 1 = (x+1)(x-1)$
- $4 - y^2 = (2-y)(2+y)$
- $(z+5)^2 = z^2 + 10z + 25$
- $(a+2b)^2 = a^2 + 4ab + 4b^2$
- $x^3 - 27 = (x-3)(x^2 + 3x + 9)$
- $y^3 + 64 = (y+4)(y^2 - 4y + 16)$
- $e^{2x} - 1 = (e^x - 1)(e^x + 1)$

1.5.1 High-lo Games to Solve Expressions

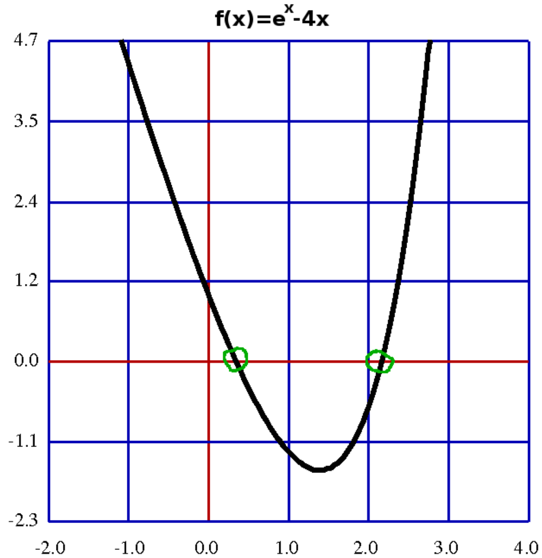
A really annoying fact is that almost every expression of the form $f(x) = 0$ you can write down *cannot* be solved for x using algebra. Your math courses, of course, tend to emphasize those expressions that can be solved and useful expressions are much more likely to be solvable by algebra than random ones. Think, for example, about how many natural laws are quadratic: the inverse-square law for gravity or electrical fields, for example, or area laws for the cross section of pipes. In this section we will give a method for getting an approximate solution for $f(x) = 0$ that can be used on any expression where there is an answer. We will give the method, first, as an example.

Example 1.39 Problem: *Get the largest possible approximate solution to $e^x = 4x$.*

First, we turn the problem into something equal to zero:

$$e^x - 4x = 0$$

and then we turn the problem into one of finding roots of a function $f(x) = e^x - 4x$. Look at the graph of the function:



Notice that there are two solutions, one between 0 and 1 and one between 2 and 3. We want the larger solution - so we start with $x = 2$ and $x = 3$ and play the following game, in which we sneak up on the value of the root from both sides by plugging values into $f(x)$.

$f(2) \cong -0.6109439$	*Lower endpoint is negative.*
$f(3) \cong 8.0855369$	*Upper endpoint is positive.*
$f(2.5) \cong 2.182494$	*Positive:* $2 \leq x \leq 2.5$
$f(2.2) \cong 0.2250135$	*Positive:* $2 \leq x \leq 2.2$
$f(2.1) \cong -0.23383009$	*Negative:* $2.1 \leq x \leq 2.2$
$f(2.15) \cong -0.015141603$	*Negative:* $2.15 \leq x \leq 2.2$
$f(2.17) \cong 0.078284041$	*Positive:* $2.15 \leq x \leq 2.17$
$f(2.16) \cong 0.031137658$	*Positive:* $2.15 \leq x \leq 2.16$
$f(2.155) \cong 0.007890179$	*Positive:* $2.15 \leq x \leq 2.155$
$f(2.153) \cong -0.001348357$	*Positive:* $2.153 \leq x \leq 2.155$
$f(2.154) \cong 0.0032666013$	*Positive:* $2.153 \leq x \leq 2.154$

At this point, since $f(2.153)$ is closer to zero than $f(2.154)$ we will accept $x = 2.153$ as our approximate answer to $e^x = 4x$.

We will now make the process in the previous example rigorous as an algorithm.

Algorithm 1.3 The high-lo game
The goal is to find a solution to the expression $f(x) = 0$.

1. *First find, by graphing or trial and error a and b so that $a < b$ and $f(a)$ and $f(b)$ have different signs. These are your initial endpoints of an interval*

2. *Pick a point c between the current endpoints of the interval containing a solution.*

3. *Compute $f(c)$.*

4. *If $f(c)$ has the same sign as the value of $f(x)$ on the left endpoint, c replaces the left endpoint; otherwise it will have the same sign as the value of $f(x)$ on the right endpoint and c replaces the right endpoint.*

5. *If the new endpoint is close enough to zero (depends on what you're doing) you are done. Otherwise go to Step 2.*

When the high-lo game is programmed into a computer, Step 2 simply chooses c to be in the middle of the interval. A human being with a graph can often chop off more than half the interval by choosing c well. Choice of c in the middle of the interval also cause the decimals of the endpoints to build up really fast - picking rounder numbers can make the process easier for a human running the high-lo game with a hand calculator.

Many calculators have the ability to run a high-lo game, or a more sophisticated root-finding algorithm. The high-lo game lets you solve expressions with a really cheap calculator that is mostly for balancing a checking account. If your calculator has a "solve" or "find roots" function it is a good idea to learn to use it.

Exercises

Exercise 1.67 *Factor the following polynomials. The root-factor theorem will help.*

a) $x^3 + x^2 - 10x + 8$. **b)** $x^3 - 2x^2 - 4x + 8$. **c)** $x^3 - 2x^2 - 6$. **d)** $2x^3 + 7x^2 + 7x + 2$. **e)** $4x^3 - 8x^2 - x + 2$. **f)** $x^3 + 3x^2x + 3$.

Exercise 1.68 *Solve the following equations.*

a) $x^3 - 4x^2 + x + 6 = 0$. **b)** $x^3 - x^2 - 3x + 2 = 0$. **c)** $x^3 - 27 = 0$. **d)** $x^3 + 3x^2 - x - 3 = 0$. **e)** $x^4 - 16 = 0$. **f)** $x^3 - 3x^2 + 3x - 1 = 0$.

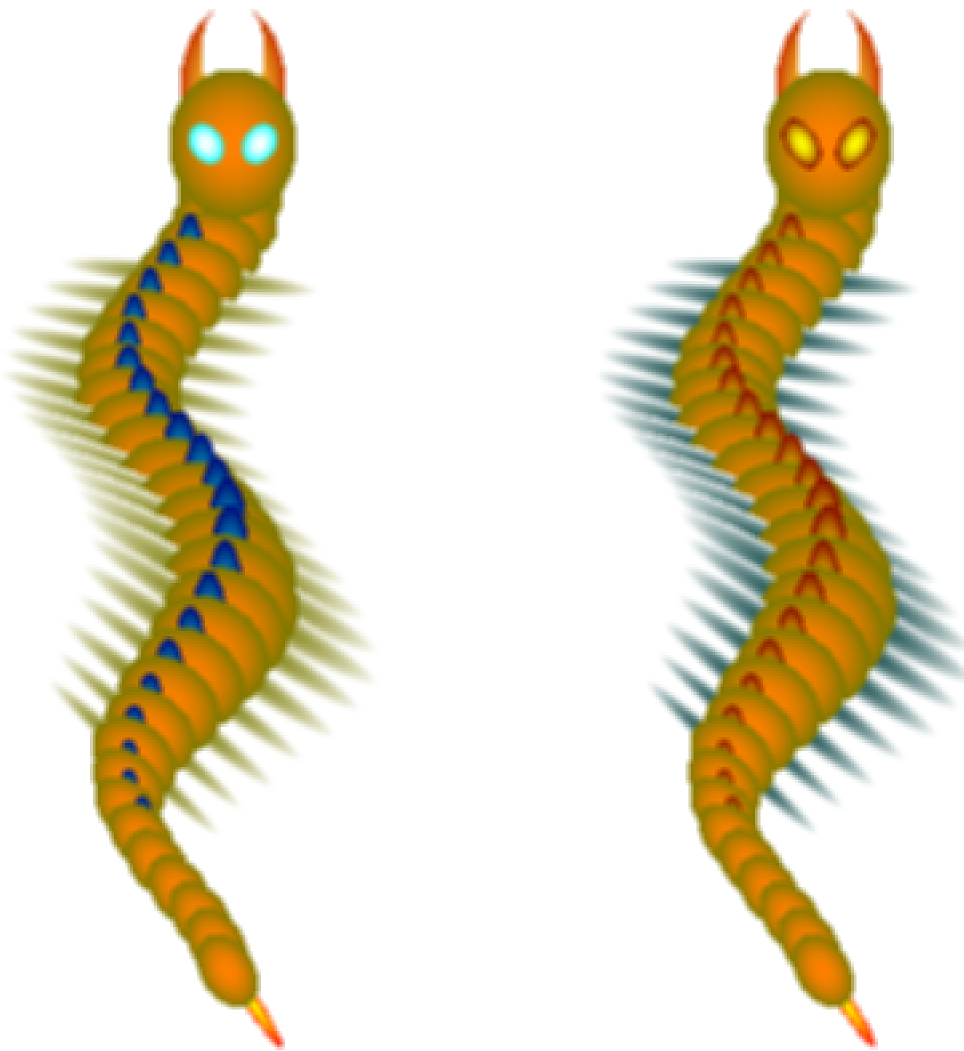
Exercise 1.69 *Using a hi-lo game solve $4ln(x) = x$. There are two solutions: approximate them both to three decimals.*

Exercise 1.70 *Using a hi-lo game solve $exp(x) = x^2 + 4$. There is one solution, approximate it to 3 decimals.*

Exercise 1.71 *Using a hi-lo game solve $x^3 + x^2 + 2x - 6 = 0$. There is one solution: approximate it to 3 decimals.*

Exercise 1.72 *Demonstrate logically that $f(x) = ax^3 + bx^2 + cx + d = 0$ must have at least one solution. Hint: what does the graph of functions of this form look like?*

Still Alive?



Chapter 2

Sequences, Series, and Limits

In this chapter we will introduce the mathematical constructs *sequences* and *series* and, in order to deal with them, we will introduce the first version of one of the central tools of advanced mathematics, the *limit*. Sequences are (possibly infinite) ordered lists of numbers. Series are obtained by adding up the numbers in a sequence in the order in which they appear in the sequence. Limits are a tool for dealing with infinite lists or sums; they may let us add up an infinite number of numbers or convince us it is impossible, depending on the membership of the list. We will focus on geometric series which have a number of applications in financial mathematics.

Lending, interest, and related financial instruments are key to making a modern economy work. Interestingly, loans and interest are very old issues in human civilization. Interest or excessive interest (called *usury*) is strongly condemned by Roman Law and many major religions. Even the immortal bard weighed in on the issue through his character Polonius. The first serious text on financial mathematics was published in 1613 demonstrating that finance is one of the founding applications of the mathematical sciences. It contained tables that pre-computed compound interest at various rates of interest, a useful tool in a society without out modern access to machine computation.

Take no usury or interest from him; but fear your God, that your brother may live with you. You shall not lend him your money for usury, nor lend him your food at a profit.
–Leviticus 25:36-37.

O ye who believe! Devour not usury, doubling and quadrupling your lendings. Observe your duty to Allah, that ye may be successful.
–Qur'an 3:130.

Neither a borrower nor a lender be, For loan oft loses both itself and friend, And borrowing dulls the edge of husbandry.
–Hamlet Act 1, scene 3.

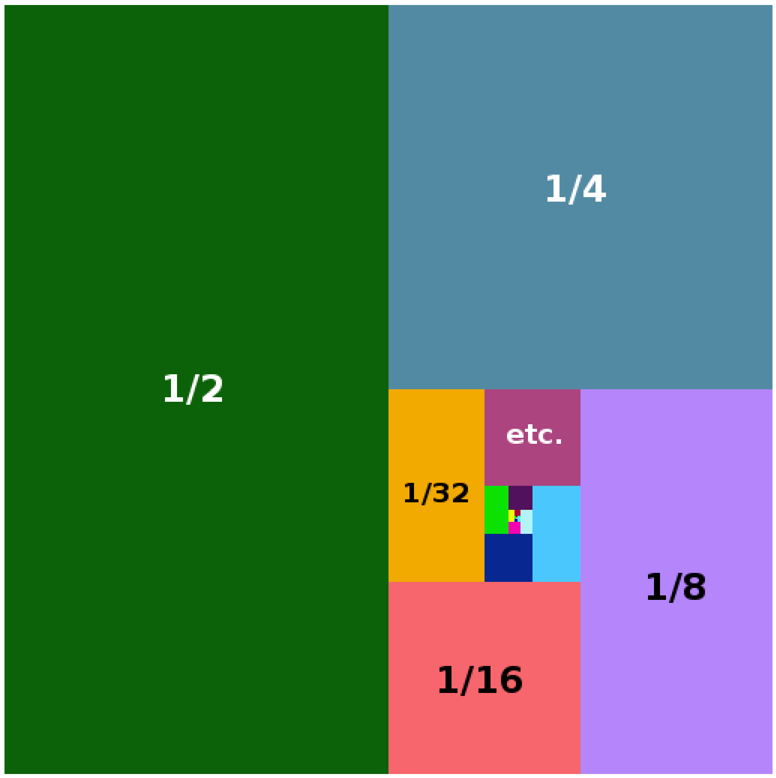
Arithmeticall Questions

Touching the buying or exchange of annuities; taking of leases for fines, or yearly rent; purchase of fee-simples; dealing for present or future possessions; and other bargaines and accounts, wherein allowance for disbursing or forbeareance of money is intended; briefly resolued, by means of certain breuiats, calculated by R.W. of London, practitioner in the arte of numbers. Examined also and corrected at the presse, by the author himselfe.

(*The first book dealing with interest and other financial mathematics, by Richard Witt, published in 1623.)*

2.1 What are Sequences and Series?

Consider the following picture, created by coloring a square with a side length of one:



The colored regions divide the square into pieces, each half the size of the one before. The five largest regions have been annotated with their respective fractions of the total area of the square. Note that:

1. Every point in the square belongs to one of the colored regions, and
2. no point belongs to two different colored regions.

From this we deduce that the sum of the areas of the regions equals the area of the square. This means that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1$$

This is a bit problematic because the sum on the left goes on forever as indicated by the use of ellipses ("$\cdots$"). The picture demonstrates that we can add up an infinite list of numbers and get a finite total. Keep this example in mind as we develop the techniques needed to work with sequences and series.

Definition 2.1 *A* **sequence** *is a list of numbers. If the list is finite:*

$$\{3, 7, 11, 15, 19, 23\}$$

then we say the sequence is **finite**. *If the list is infinite:*

$$\left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \ldots\right\}$$

Then we say the sequence is **infinite**.

Example 2.1 *Since ellipsis notation can be ambiguous sometimes we have a more precise way of specifying infinite sequences. The second example in the definition above is supposed to be all positive, whole-number powers of* $\frac{1}{3}$. *We introduce a whole number parameter* n *and write*

$$\left\{ \left(\frac{1}{3} \right)^n \right\}_{n=1}^{\infty}$$

The $n = 1$ *and* ∞ *at the top and bottom of the right hand bracket indicate which values* n *can take on; all values one or more, but by convention not* ∞ *itself, which is not a number.*

This notation can also be used for finite sequences. For example

$$\{2^n\}_{n=0}^{10}$$

is a shorter way to write $\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\}$. *Even though it is finite you probably would not want to write*

$$\left\{ \left(\frac{3}{4} \right)^n \right\}_{n=1}^{100}$$

as an explicit list. The final term of this series is:

$$\frac{515377520732011331036461129765621272702107522001}{1606938044258990275541962092341162602522202993782792835301376}$$

which is sort of long.

Our definition of series is based on our definition of sequences. A series is the result of adding up a sequence.

Example 2.2 *The object:*

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

is a sequence while

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

is a series. *This series has a sum or value of 55.*

Definition 2.2 *A* **series** *is a list of numbers that are to be added up in the order given. If the list is finite:*

$$3 + 7 + 11 + 15 + 19 + 23$$

then we say the series is **finite**. *If the list is infinite:*

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$$

Then we say the series is **infinite**. *In the next section we will learn how to tell that the infinite series in this example has a sum of* $\frac{1}{2}$.

Finite sums and series are just notational innovations, an increase in the vocabulary of our mathematical language. The infinite versions of these objects, however, can be well behaved or badly behaved. This notion of good or bad behavior requires the idea of a *limit*.

Definition 2.3 *Suppose that* $S = \{a_1, a_2, a_3, \ldots\}$ *is an infinite sequence. Then we say the number* L *is the* **limit** *of the sequence* S *if for every positive number* ϵ *we can find a number* N *(which can be different for different values of* ϵ*) so that all terms* a_i*, for which* $i \geq N$*, differ from* L *by less than* ϵ.

Informally, the number L *is the limit of the sequence* S *if the members of* S *eventually get and stay closer to* L *than any fixed, positive distance.*

Example 2.3 *The limit of*

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

is 0. To see this notice that for any positive number ϵ *we can round* $\frac{1}{\epsilon}$ *up to the nearest integer* N *to find* N *so that* $\frac{1}{N}$, $\frac{1}{N+1}$, $\frac{1}{N+2}$, $\ldots$ *are all smaller than epsilon and therefore closer to zero than* ϵ.

Remember that the distance between two numbers a and b is $|a-b|$, the absolute value of their difference.

An even more informal way to think of the limit of a sequence is this. If the sequence is approaching a value L then that number is the limit. If the sequence fails to approach any single number L then it does not have a limit; we might also say the limit of such a sequence is undefined.

There are three common behaviors for a sequence:

1. Approach a limit L.
2. Jump around without settling anywhere.
3. Grow toward ∞ (or $-\infty$).

Example 2.4 Examples of the three types of sequence behavior

1. *The sequence*

$$\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$$

takes on the values $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\}$. *Since the top and bottom of the fractions are getting closer and closer it would not be too hard to show that* $L = 1$ *is the limit of this series.*

2. *The sequence*

$$\{(-1)^n\}_{n=0}^{\infty}$$

takes on the values $\{1, -1, 1, -1, 1, -1, 1, -1, \ldots\}$. *Since the values 1 and -1 both occur infinitely often, adjacent terms of the sequence are a distance of two apart infinitely often. This makes it impossible for the sequence to approach a value L; the sequence jumps around.*

3. *The sequence*

$$\{2^n\}_{n=0}^{\infty}$$

takes on the values $\{1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots\}$. *Since the values grow larger indefinitely this sequence diverges to infinity. We can say that the limit of this sequence is infinity, but this is another way of saying the sequence fails to have a limit: again,* ∞ *is not a number.*

To show that a sequence diverges to infinity you must show that it eventually gets larger than any finite number. For the third sequence above this is easy. For a constant c *let* n *be the next whole number larger than* $\log_2(c)$. *Then* 2^n, *which is a member of the sequence, must be bigger than* c *and we have demonstrated divergence.*

Now that we have sequences at least somewhat under control, we are ready to extend our tools to series. Oddly, the extension is performed by constructing a sequence from the series we are interested in.

Definition 2.4 *If* R *is the series* $a_1 + a_2 + a_3 + \cdots$ *then* **the sequence of partial sums** *of* R *is the sequence*

$$\{a_1, (a_1 + a_2), (a_1 + a_2 + a_3), (a_1 + a_2 + a_3 + a_4), \ldots\}$$

The nth term of the sequence of partial sums for a series R *is the sum of the first* n *terms of* R.

Example 2.5 Partial sums of powers of 1/2
Suppose R is the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$$

which we already know adds to 1. The sequence of partial sums is:

$$\left\{\frac{1}{2},\left(\frac{1}{2}+\frac{1}{4}\right),\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right),\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right),\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\right),\ldots\right\}$$

If we do the addition we get

$$\left\{\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\frac{31}{32}\right\}$$

From this we see the general form of the sequence of partial sums is

$$\left\{\frac{2^n-1}{2^n}\right\}_{n=0}^{\infty}$$

The more impatient among the readers may be wondering why we went to all the trouble to define the sequence of partial sums. The answer is that the whole notion of a series converging depends on its sequence of partial sums.

Definition 2.5 Convergence of a series
A series has a sum L if and only if its sequence of partial sums converges to L as a limit. In this case we say the series **converges to L**.

It is not hard to see that the limit of $\left\{\frac{2^n-1}{2^n}\right\}_{n=0}^{\infty}$ is 1. Since this is the sequence of partial sums of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$ from Example 2.5 this limit demonstrates that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots=1$$

using the definition of convergence of a series.

We are now ready to define some types of sequences that are especially easy to deal with.

Definition 2.6 Monotone sequences.

A sequence $S=\{a_n\}_{n=0}^{\infty}$ is **increasing** *if $a_n < a_{n+1}$ for all n.*

A sequence $R=\{b_n\}_{n=0}^{\infty}$ is **decreasing** *if $b_n > b_{n+1}$ for all n.*

A sequence that is either increasing or decreasing is said to be **monotone**.

Definition 2.7 Bounded sequences.
A sequence $S=\{a_n\}_{n=0}^{\infty}$ is **bounded above** *if there is a fixed constant c so that $c > a_n$ for all n.*

We say S is **bounded below** *if we can find a constant d so that $d < a_n$ for all n.*

A sequence that is bounded above and below is called a **bounded** *sequence.*

Bounded sequences are prevented, by their bounds, from diverging to one, the other, or both infinities. Look at our favorite series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$$

The sequence of partial sums grows, but by a smaller amount each time, and so the sequence increases but, since it totals one, is bounded above. This phenomenon happens fairly often and it is formalized in the following fact.

Fact 2.1 Bounded, monotone sequences

This fact has three versions:

1. *A bounded, monotone sequence converges to a limit.*
2. *An increasing sequence that is bounded above converges to a limit.*
3. *A decreasing sequence that is bounded below converges to a limit.*

Let us revisit the sequence of partial sums of the powers of 1/2:

$$\left\{\frac{2^n - 1}{2^n}\right\}_{n=0}^{\infty}$$

Notice that the sequence grows at each step (is increasing) but never grows beyond 1. It is thus an example of an increasing sequence that is bounded above (by 1). It is sometimes possible to prove that a sequence has a limit, even when we cannot compute it.

Example 2.6 Using the monotone sequence fact.
Problem: *Demonstrate that the sequence*

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

has a limit.

Strategy: Prove the sequence is decreasing and find a lower bound. Notice that this sequence has the values $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$. *Since* $1, 2, 3, 4, 5, \ldots$ *are getting bigger, it is clear their reciprocals are getting smaller. This means that the sequence is decreasing. Since every term of the sequence is positive, it follows that the sequence is bounded below by zero. This means that the sequence is a decreasing sequence that is bounded below and so, by the fact, has a limit.*

Since the terms of the sequence get as close to zero as you like, it is easy to see that the limit is zero.

Definition 2.8 Mathematical notation for limits.
If a sequence $S = \{a_1, a_2, a_3, \ldots\}$ *has the number* L *as a limit, then the mathematical notation for this situation is*

$$\underset{n \to \infty}{Lim} a_n = L$$

Example 2.7 Using limit notation. *Drawing on early examples in this section:*

$$\underset{n \to \infty}{Lim} \left(\frac{n}{n+1}\right) = 1$$

$$\underset{n \to \infty}{Lim} \left(\frac{1}{n}\right) = 0$$

One of the more useful parts of Chapter 1 was that it listed algebraic properties of the operators and functions that it was explaining. It turns out that limits have a number of useful properties as well. These properties can be used to work limit problems.

Fact 2.2 Algebraic properties of limits.
Suppose that $S = \{a_1, a_2, a_3, \ldots\}$, $R = \{b_1, b_2, b_3, \ldots\}$ *are both sequences that have limits as follows:*

$$\mathop{Lim}_{n \to \infty} a_n = L \qquad \mathop{Lim}_{n \to \infty} b_n = M$$

If c *is a constant then:*

1. $\mathop{Lim}_{n\to\infty} (c \cdot a_n) = c \cdot L$
2. $\mathop{Lim}_{n\to\infty} (a_n + b_n) = L + M$
3. $\mathop{Lim}_{n\to\infty} (a_n - b_n) = L - M$

In other words, constant multiples, sums, and differences of sequences that have limits are also sequences that have limits.

Example 2.8 Using the algebraic properties of limits.
We already know the following pairs of limits:

$$\mathop{Lim}_{n \to \infty} \left(\frac{n}{n+1}\right) = 1 \qquad \mathop{Lim}_{n \to \infty} \left(\frac{1}{n}\right) = 0$$

These imply, by using the algebraic properties of limits, that

$$\mathop{Lim}_{n \to \infty} \left(\frac{3n}{n+1}\right) = 3 \times \mathop{Lim}_{n \to \infty} \left(\frac{n}{n+1}\right) = 3 \times 1 = 3$$

$$\mathop{Lim}_{n \to \infty} \left(\frac{1}{n} + \frac{n}{n+1}\right) = \mathop{Lim}_{n \to \infty} \left(\frac{1}{n}\right) + \mathop{Lim}_{n \to \infty} \left(\frac{n}{n+1}\right) = 0 + 1 = 1$$

In order to have a limit, a series must have a sequence of partial sums that converges. That, in turn, means that the terms of the series have to get very small (in absolute value) as n grows or they will mess up the convergence. A more precise statement of this is the following.

Fact 2.3 A test for failure to converge.
Suppose that $a_1 + a_2 + a_3 + \cdots$ *is a series. If the sequence*

$$\{a_1, a_2, a_3, \ldots\}$$

does not converge to zero then the series itself does not have limit (its sum is undefined).

WARNING! The fact above is *one sided*. If the sequence *does* converge to zero then the series may converge but it also may diverge.

Definition 2.9 general term of a series
In a series $a_1 + a_2 + a_3 + \cdots$ *we call* a_n *the* **general term** *of the series.*

Another way to state the fact that permits us to test for failure of a series to converge is: "If a series converges then the corresponding sequence of general terms must converge to zero." In order to reinforce the warning we now give an example of a series that fails to converge in spite of the fact its sequence of general terms does converge to zero.

Example 2.9 A divergent series whose general term converges to zero.
The series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

does not have a limit. To see this we will replace the terms as follows:

$$1 \quad \tfrac{1}{2} \quad \tfrac{1}{3} \quad \tfrac{1}{4} \quad \tfrac{1}{5} \quad \tfrac{1}{6} \quad \tfrac{1}{7} \quad \tfrac{1}{8} \quad \tfrac{1}{9} \quad \tfrac{1}{10} \quad \tfrac{1}{11} \quad \tfrac{1}{12} \quad \tfrac{1}{13} \quad \tfrac{1}{14} \quad \tfrac{1}{15} \quad \tfrac{1}{16} \quad \cdots$$

are replaced with

$$1 \quad \tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{4} \quad \tfrac{1}{8} \quad \tfrac{1}{8} \quad \tfrac{1}{8} \quad \tfrac{1}{8} \quad \tfrac{1}{16} \quad \tfrac{1}{16} \quad \tfrac{1}{16} \quad \tfrac{1}{16} \quad \tfrac{1}{16} \quad \tfrac{1}{16} \quad \tfrac{1}{16} \quad \tfrac{1}{16} \quad \cdots$$

The new terms are less than or equal to the original ones and all the terms are positive. This means that the sum of the new terms cannot be larger than the sum of the old terms.

Group the terms of the new sequence:

$$1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)+\cdots$$

$$=1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{8}{16}+\cdots$$

$$=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

Which means the second sequence is one plus an infinite sum of one-halfs. This will eventually grow past any finite number and so diverges to infinity. The original sequence therefore has a sum that is at least infinity; it too must diverge to infinity and so fails to have a limit. The sum of reciprocals is a famous series with its own name. It is called the **harmonic series**.

The trick used to demonstrate that the harmonic series does not have a limit uses a useful and intuitive fact that we will now state explicitly.

Fact 2.4 The comparison tests.

Suppose that $a_1+a_2+a_3+\cdots$ *converges to a limit* L *and that* $a_n \geq 0$ *for all* n*. Then if* $0 \leq b_n \leq a_n$ *we may deduce that* $b_1+b_2+b_3+\cdots$ *also converges to a limit that is no larger than* L*.*

On the other hand if $a_1+a_2+a_3+\cdots$ *diverges to* ∞ *and we have* $b_n \geq a_n$ *for all* n *then* $b_1+b_2+b_3+\cdots$ *must also diverge to infinity.*

The "On the other hand" version of this fact is the one used to demonstrate the divergence of the harmonic series. The two halves of the fact above may be stated informally as follows. A series of positive terms that are all smaller, term-by-term, than a series that has a limit must, itself have a limit. A series of terms that are larger, term-by-term, than a series that diverges to infinity must also diverge to infinity.

Example 2.10 Demonstrating divergence by comparison

Problem: *Show that*

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\cdots$$

diverges to infinity.

Strategy: Use comparison to the harmonic series. Notice that if $n \geq 2$ that taking the square-root makes the number smaller. This means that $n > \sqrt{n}$ and so their reciprocals share the opposite relationship:

$$\frac{1}{n} < \frac{1}{\sqrt{n}}$$

This means that the series of sums of reciprocals of square roots is larger than the series of sums of reciprocals (which is the harmonic series). Since we have already demonstrated that the harmonic series diverges to infinity, comparison permits us to deduce that the sequence in this example also diverges to infinity.

At this point we introduce **sigma notation** or **summation notation** for working with series.

Definition 2.10 *So far we have used explicit sums for series. The symbol $\sum$ means "add up". It can be used with finite or infinite series. The finite series*

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

becomes

$$\sum_{n=1}^{10} n$$

The variable n is called the **index** *of the sum and its limits are given at the top and bottom of the sum-symbol. The infinite series*

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$$

becomes

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

If a series has a limit we say it *converges*, otherwise we say it *diverges*. In order for comparison to be a useful technique for demonstrating convergence or divergence we need to have many sequences available whose divergence or convergence behavior we already know. The following fact is handy in this regard.

Fact 2.5 ***p*-series**
If p is a real number then the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for all $p > 1$ and diverges for all $p \leq 1$.

Notice this fact, with $p = 1$, includes the harmonic series. Demonstrating that this fact is true for $p \neq 1$ requires tools that we will develop in Chapter 5, but we can start using it now. Notice that p-series make Example 2.10 much simpler. The sequence in that example is just a p-series for $p = 1/2$ and so obviously diverges (if you have the p-series fact).

Exercises

Exercise 2.1 *Write as an explicit list, using ellipses if you need to, the following sequences. For infinite series use at least four numerical terms.*

a) $\left\{n^2+1\right\}_{n=1}^{6}$. **b)** $\left\{\frac{n+2}{n+3}\right\}_{n=0}^{7}$. **c)** $\left\{3^n - 2^n\right\}_{n=0}^{5}$. **d)** $\left\{\frac{2^n}{3^n}\right\}_{n=1}^{\infty}$. **e)** $\left\{\left(\frac{1}{4}\right)^n\right\}_{n=0}^{\infty}$. **f)** $\left\{\frac{n^2}{n^2+1}\right\}_{n=0}^{\infty}$.

Exercise 2.2 *Use curly brace notation to write the following sequences. You will need to find the pattern to make a formula. Finding a formula for (**f**) is possible but very difficult. You may* explain *the pattern instead if you wish.*

a) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

b) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

c) $\{1, 3, 5, 7, 9, 11, \ldots\}$

d) $\{2, 3, 5, 9, 17, 33, 65, 128, 257, \ldots\}$

e) $\{1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, \ldots\}$

f) $\left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \ldots\right\}$

Exercise 2.3 *For each of the sequences in Exercises 2.1 and 2.2 state if the sequence is finite or infinite.*

Exercise 2.4 *For each of the following infinite series state which of the three behaviors mentioned in this section the series exhibits and, if it converges to a limit, find the limit.*

a) $\left\{\left(\frac{1}{3}\right)^n\right\}_{n=0}^{\infty}$. **b)** $\left\{\left(-\frac{1}{2}\right)^n\right\}_{n=0}^{\infty}$. **c)** $\{\sqrt[3]{n}\}_{n=1}^{\infty}$. **d)** $\left\{n^{-2}\right\}_{n=1}^{\infty}$. **e)** $\{e^n\}_{n=2}^{\infty}$.

f) $\{ln(n)\}_{n=2}^{\infty}$. **g)** $\{1 + (-1)^n\}_{n=0}^{\infty}$. **h)** $\left\{\frac{(-2)^n}{1-2^n}\right\}_{n=0}^{\infty}$. **i)** $\left\{\frac{n-1}{n+1}\right\}_{n=1}^{\infty}$. **j)** $\left\{\frac{3n+5}{1+n}\right\}_{n=0}^{\infty}$.

Exercise 2.5 *For each of the following series, give the first six members of the sequence of partial sums.*

a) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10.$

b) $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \frac{1}{4096} + \cdots.$

c) $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 11 - 12.$

d) $\sum_{n=0}^{\infty} \frac{3}{2^n}$. **e)** $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$. **f)** $\sum_{n=1}^{10} \frac{1}{n}$.

Exercise 2.6 *For each of the following infinite series state if the series has a limit or not and give a reason why.*

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$. **b)** $\sum_{n=1}^{\infty} \frac{1}{n^2}$. **c)** $\sum_{n=0}^{\infty} \frac{n}{n+1}$. **d)** $\sum_{n=0}^{\infty} \frac{5}{2^n}$. **e)** $\sum_{n=1}^{\infty} \frac{1}{n} + 1$. **f)** $\sum_{n=1}^{\infty} \frac{n^2}{n+1}$.

Exercise 2.7 *Compute the sum of the following series using examples in this section and the algebraic properties of limits. Some of the series diverge, others have finite sums.*

a) $\sum_{n=1}^{\infty} \frac{3}{2^n}$. **b)** $\sum_{n=1}^{\infty} \frac{5}{2^n}$. **c)** $\sum_{n=0}^{\infty} \frac{7n}{n+1}$. **d)** $\sum_{n=1}^{\infty} \frac{9}{n}$. **e)** $\sum_{n=1}^{\infty} \frac{3n}{n+1} - \frac{1}{2^n}$. **f)** $\sum_{n=1}^{\infty} \frac{3}{2^n} - \frac{4n}{2n+2}$.

Exercise 2.8 *If*

$$\sum_{0}^{\infty} a_n = 2 \qquad \text{and} \qquad \sum_{0}^{\infty} b_n = 3$$

compute the limits of the following series. Remember to simplify your answers.

a) $\sum_{n=0}^{\infty} a_+ b_n$. **b)** $\sum_{n=0}^{\infty} a_n - b_n$. **c)** $\sum_{n=0}^{\infty} 2a_n$. **d)** $\sum_{n=0}^{\infty} \frac{1}{3} b_n$. **e)** $\sum_{n=0}^{\infty} 7a_n - 3b_n$. **f)** $\sum_{n=0}^{\infty} \sqrt{2}a_n - \sqrt{3}b_n$.

Exercise 2.9 *For each of the following series, demonstrate by comparison that the series converges or diverges. Do not compute the limit of the convergent series. You may use the results of Exercise 2.15 if you wish.*

a) $\sum_{n=1}^{\infty} \frac{2}{n}$. **b)** $\sum_{n=2}^{\infty} \frac{1}{n-1}$. **c)** $\sum_{n=1}^{\infty} \frac{1}{3n^2}$. **d)** $\sum_{n=1}^{\infty} \frac{1}{2n^3}$. **e)** $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$. **f)** $\sum_{n=2}^{\infty} \frac{1}{n^2-n}$.

Exercise 2.10 *Using the definition of a limit demonstrate that*

$$\begin{matrix} Lim \\ n \to \infty \end{matrix} \frac{1}{n^2} = 0$$

Exercise 2.11 *Using the definition of a limit demonstrate that*

$$\begin{matrix} Lim \\ n \to \infty \end{matrix} \frac{2n^2}{n^2+1} = 2$$

Exercise 2.12 *Demonstrate that the sequence*

$$\{ln(n)\}_{n=2}^{\infty}$$

diverges to infinity.

Exercise 2.13 *Demonstrate that the sequence*

$$\left\{\frac{n^2}{n+1}\right\}_{n=2}^{\infty}$$

diverges to infinity.

Exercise 2.14 *If* $S = \{a_1, a_2, a_3, \ldots\}$ *is a bounded sequence which is not monotone* but *the sequence* $\{a_k, a_{k+1}, a_{k+2}, \ldots\}$ *is monotone for some positive whole number* k*, demonstrate logically that* S *has a limit.*

Exercise 2.15 *Demonstrate logically that the two series*

$$\sum_{n=0}^{\infty} a_n \quad or \quad \sum_{n=k}^{\infty} a_n$$

for $k \geq 0$ *either both converge or both diverge.*

Exercise 2.16 *The colored partitioned square near the beginning of this section was created using the following steps.*

1. *Color half the remaining space from the right.*
2. *Color half the remaining space from the top.*
3. *Color half the remaining space from the right.*
4. *Color half the remaining space from the bottom.*
5. *Go back to step 1.*

The steps continue forever, but we stop drawing the picture after filling 16 areas, because after that you cannot see the areas. This square corresponded to the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

Question: which sequence does the picture correspond to if, instead, we color one-third of the remaining space in steps 1-4?

Exercise 2.17 *Draw the picture, based on coloring one-third of the remaining area, described in Exercise 2.16.*

Exercise 2.18 *Find the next three terms of the sequence* $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ *assuming it follows a simple rule.*

Exercise 2.19 *Find the next three terms of the sequence* $1, 1, 1, 3, 5, 9, 17, 31, 57, \ldots$ *assuming it follows a simple rule.*

Exercise 2.20 *Find, to four decimals,*

$$\underset{n \to \infty}{Lim} \frac{a_{n+1}}{a_n}$$

if a_n *is the sequence from Exercise 2.18*

Exercise 2.21 *Can you find a simple mathematical rule or algorithm for the following finite sequence?*

{27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1}

Exercise 2.22 *Compute the finite sequence like that in Exercise 2.21 that starts with 38, instead of 27. It ends at 1.*

2.2 Geometric Series

Geometric series occur whenever adjacent terms of a series have the same ratio. The sequence

$$1 = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

that we used extensively as an example in the last section is a geometric series: the ratio of adjacent terms is always 1/2. We are now ready for a formal definition.

Definition 2.11 *A* **geometric series** *is any series where the ratio* r *of adjacent terms is constant. Such series always have the form*

$$\sum_{n=0}^{m} d \cdot r^n$$

if they are finite and

$$\sum_{n=0}^{\infty} d \cdot r^n$$

if they are infinite. The number d *is the* **first term** *of the sequence while* r *is called the* **ratio** *of the sequence.*

Example 2.11 Testing if a series is geometric.
Problem: *which of the following series are geometric?*

1. $\sum_{n=1}^{\infty} \frac{1}{n}$.
2. $\sum_{n=0}^{\infty} \frac{3}{4^n}$.
3. $\sum_{n=2}^{\infty} \frac{2^n}{5^{2n}}$.

Strategy: See if the ratio of adjacent terms $\frac{a_{n+1}}{a_n}$ *is constant.*

1. *The ratio of adjacent terms is*

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}$$

which is not constant; the sequence is not geometric.

2. *The ratio of adjacent terms is*

$$\frac{\frac{3}{4^{n+1}}}{\frac{3}{4^n}} = \frac{3\cdot 4^n}{3\cdot 4^{n+1}} = \frac{1}{4}$$

which is constant; the sequence is geometric.

3. *The ratio of adjacent terms is*

$$\frac{\frac{2^{n+1}}{5^{2(n+1)}}\ \frac{2^n}{5^{2n}}}{=} \frac{2^{n+1}\cdot 5^{2n}}{2^{n+1}\cdot 5^{2n+2}} = \frac{2}{5^2} = \frac{2}{25}$$

which is constant; the sequence is geometric.

We are interested in geometric series for two reasons. First, they solve a number of problems in financial mathematics. Second, there is an exact formula. In order to get the exact formula we need to return briefly to the land of algebra. A particular form of one of our algebra facts says

$$(1-x)(1+x) = 1-x^2$$

Let's generalize this.

$$\begin{aligned}(1-x)(1+x+x^2) &= \big((1+x+x^2) - x(1+x+x^2)\big) \\ &= \big(1+x+x^2-x-x^2-x^3\big) \\ &= 1-x^3\end{aligned}$$

Notice that all the middle terms canceled out. Does this happen in the next version of the expression?

$$\begin{aligned}(1-x)(1+x+x^2+x^3) &= \big((1+x+x^2+x^3) - x(1+x+x^2+x^3)\big) \\ &= \big(1+x+x^2+x^3-x-x^2-x^3-x^4\big) \\ &= 1-x^4\end{aligned}$$

In fact it is always the case that

$$(1-x)(1+x+x^2+\cdots+x^k) = 1-x^{k+1}$$

because all the middle terms cancel out. This expression is the key to finding an exact formula for finite geometric series.

Fact 2.6 Finite geometric series formula.
We know that

$$(1-x)(1+x+x^2+\cdots+x^k) = (1-x^{k+1})$$

If we divide by $(1-x)$ we get

$$1+x+x^2+\cdots+x^k = \frac{1-x^{k+1}}{1-x}$$

If we let $x = r$ and we get

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{k+1}}{1 - r}$$

Multiply both sides by d and we get

$$d + dr + dr^2 + \cdots + dr^k = d\frac{1 - r^{k+1}}{1 - r}$$

Converting the left hand side to sigma notation we obtain the classical formula for geometric series:

$$\sum_{n=0}^{k} d \cdot r^n = d \cdot \frac{1 - r^{k+1}}{1 - r}$$

Not that we have the formula, let's practice using it.

Example 2.12 Problem: *Compute*

$$1 + 2 + 4 + 8 + 16 + \cdots + 2^9 + 2^{10}$$

Since this is $\sum_{n=0}^{10} 2^n$ we get

$$\frac{1 - 2^{11}}{1 - 2} = \frac{1 - 2048}{1 - 2} = \frac{-2047}{-1} = 2047$$

Problem: *Compute*

$$\sum_{n=0}^{8} 2 \cdot 5^n$$

Apply the finite geometric series formula and we get:

$$2 \cdot \frac{1 - 5^9}{1 - 5} = 2 \cdot \frac{1 - 1953125}{-4} = 2 \cdot \frac{1953124}{4} = 976562$$

We are now ready to derive the infinite geometric series formula, an endeavor that will require us to use limits. We need to understand the behavior of the quantity r^k where r is the ratio of a geometric series.

Example 2.13 Behavior of powers of a constant.

Consider the following six examples of the sequence of powers of a constant c, starting with $c^0 = 1$.

Constant	*Powers*	*Behavior*
-2	*1,-2,4,-8,16,-32,...*	*Growing in both directions; diverges.*
-1	*1,-1,1,-1,1,-,1,...*	*Jumping back and forth.*
$-\frac{1}{2}$	*1,* $\frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \frac{1}{16}, \frac{-1}{32}, \ldots$	*Jumping back and forth but shrinking to zero.*
$\frac{1}{2}$	*1,* $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$	*Shrinking to zero.*
1	*1,1,1,1,1,1,...*	*Stays at one.*
2	*1,2,4,8,16,32,...*	*Grows indefinitely.*

These behaviors cover all the possibilities for constants in the five following ranges:

1. *Less than minus one: jumps around while diverging.*
2. *Equal to minus one: jumps around without converging.*
3. *Strictly between -1 and 1: converges to zero.*
4. *Equal to one: stays one.*
5. *Greater than one: grows indefinitely.*

Recall that a series must diverge if its sequence of general terms does not converge to zero. This tells us that **there is no hope an infinite geometric series will converge unless its ratio is strictly between one and negative one.**

With the result from Example 2.13 we are ready to create the infinite geometric series formula.

Fact 2.7 The infinite geometric series formula. *We start with the finite geometric series formula.*

$$d \cdot \frac{1 - r^{k+1}}{1 - r} = \sum_{n=0}^{k} d \cdot r^n$$

Take the limit as k goes to infinity of both sides:

$$\begin{matrix} Lim \\ k \to \infty \end{matrix} d \cdot \frac{1 - r^{k+1}}{1 - r} = \begin{matrix} Lim \\ k \to \infty \end{matrix} \sum_{n=0}^{k} d \cdot r^n$$

Since 1, d, and r don't involve k we can simplify the limit as follows:

$$d \cdot \frac{1 - \underset{k \to \infty}{Lim} r^{k+1}}{1 - r} = \sum_{n=0}^{\infty} d \cdot r^n$$

We already know that if $-1 < r < 1$ that $\underset{k \to \infty}{Lim} r^{k+1} = 0$ and that otherwise the limit is undefined. Therefore:

$$\frac{d}{1 - r} = \sum_{n=0}^{\infty} d \cdot r^n$$

which is the infinite geometric series formula.

Example 2.14 Infinite geometric series.
Compute the following infinite geometric series:

1. $\sum_{n=0}^{\infty} \frac{5}{2^n}$
2. $\sum_{n=0}^{\infty} \frac{1}{(-3)^n}$
3. $\sum_{n=0}^{\infty} 2q^{-n}$ *where $q > 1$ is a whole number.*
4. $\sum)_{n=0}^{\infty} 1.1^n$.

Strategy: identify the first term d and the ratio r and apply the infinite geometric series formula.

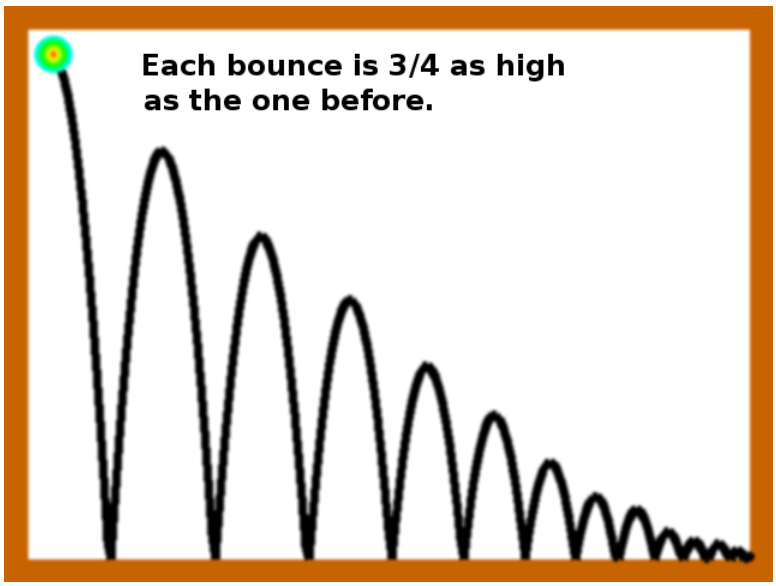
1. *$d = 5$, $r = 1/2$ so the sum is $\frac{5}{1-1/2} = 10$.*

2. $d = 1$, $r = -1/3$ so the sum is $\frac{1}{1-(-1/3)} = \frac{1}{4/3} = \frac{3}{4}$.

3. $d = 2$, $r = q^{-1} = 1/q$ so the sum is $\frac{2}{1-1/q} = \frac{2}{\frac{q-1}{q}} = \frac{2q}{q-1}$.

4. $d = 1$, $r = 1.1$ so the sum does not exist: r is not strictly between minus one and one.

A tricky series: the bouncing ball



The ball above is dropped from a height of two meters. What is the total vertical distance it travels?

The first fall is 2 meters and, after that each fall is $\frac{3}{4}$ the distance of the one before. This means the total distance the ball falls is given by the geometric series

$$2 + 2 \cdot \frac{3}{4} + 2 \cdot \frac{9}{16} + 2 \cdot \frac{27}{64} + \cdots$$

The problem is this: the ball also rises. The first rise is as high as the second drop meaning that the rising distance is given by the series

$$2 \cdot \frac{3}{4} + 2 \cdot \frac{9}{16} + 2 \cdot \frac{27}{64} + 2 \cdot \frac{81}{256} + \cdots$$

That means the total vertical distance the ball moves is the sum of these two series with $d = 2$, $r = 3/4$ and $d = 2 \cdot \frac{3}{4} = 3/2$, $r = 3/4$ yielding an answer of

$$\frac{2}{1-\frac{3}{4}} + \frac{3/2}{1-\frac{3}{4}}$$

$$= \frac{2}{1/4} + \frac{3/2}{1/4} = 2 \times 4 + \frac{3}{2} \times 4 = 8 + 6 = 14 \text{ meters}$$

A real ball, of course, does not bounce forever and so the number of bounces the ball makes isn't infinite. At the point at which the ball stops bouncing the infinite number of remaining terms of the series add up to a very small number meaning that 14 meters is quite close to the correct answer.

Exercises

Exercise 2.23 *Which of the following series are geometric? Both state the answer and give a reason.*

a) $\sum_{n=0}^{100}\left(\frac{3}{4}\right)^n$. **b)** $\sum_{n=0}^{100}\frac{7}{3^n}$. **c)** $\sum_{n=1}^{\infty}\frac{1-n}{1+n}$. **d)** $\sum_{n=1}^{\infty}\frac{n^2}{n^4+1}$. **e)** $\sum_{n=0}^{\infty}\frac{(1+1)^n}{(1+1+1)^n}$. **f)** $\sum_{n=0}^{\infty}\frac{3^n}{2^{3n}}$.

Exercise 2.24 *Compute the following finite geometric sums. Hint: recover the ratio by dividing adjacent terms if it is not obvious.*

a) $1+2+4+8+16+32+64+128+256+512+1024+\cdots+8192$. **b)** $9+27+81+243+729+\cdots+59049+117147$.

c) $1+\frac{3}{2}+\frac{9}{4}+\cdots+\frac{2187}{128}$. **d)** $1+1.1+1.1^2+1.1^3+\cdots+1.1^8$.

e) $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots-\frac{1}{19683}$. **f)** $a+ab+ab^2+ab^3+\cdots+ab^{17}$.

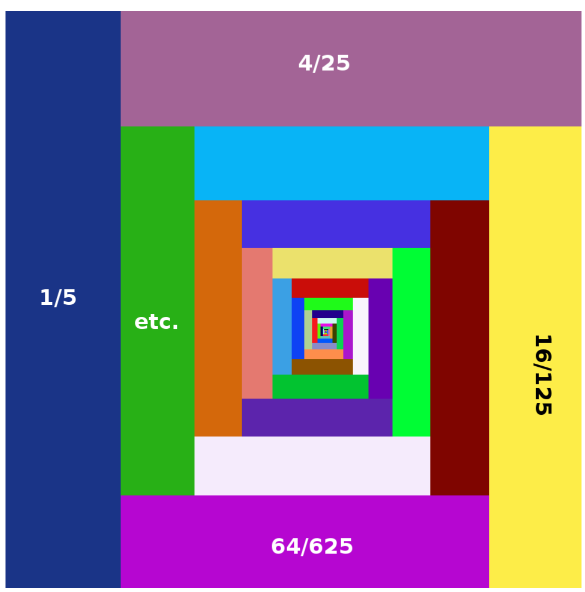
Exercise 2.25 *Compute the following infinite geometric sums. Remember that some of them may be undefined.*

a) $\sum_{n=0}^{\infty}\frac{2}{3^n}$. **b)** $\sum_{n=0}^{\infty}3\times\left(\frac{1}{2}\right)^n$. **c)** $7+7/3+7/9+7/27+7/81/+7/243+\cdots$.

d) $0.1+0.01+0.001+0.0001+0.00001+0.000001+\cdots$. **e)** $\sum_{n=0}^{\infty}\frac{2^n}{5^n}$. **f)** $\sum_{n=0}^{\infty}\frac{2^n}{(-3)^n}$.

g) $\sum_{n=0}^{\infty}3\cdot(1.02)^n$ **h)** $\sum_{n=0}^{\infty}\frac{5^n}{4^n}$. **i)** $\sum_{n=0}^{\infty}3\times(-0.9)^n$. **j)** $\sum_{n=0}^{\infty}\frac{3}{4\cdot 5^n}$.

Exercise 2.26 *Compute the sum of the series that is the answer to Exercise 2.16 in the previous section.*



Exercise 2.27 *The picture above is similar to the picture at the beginning of the first section of this chapter, but it goes by fifths rather than halves. It also gives a series that adds up to one. Figure out series in the form*

$$\sum_{n=0}^{\infty} d\cdot r^n$$

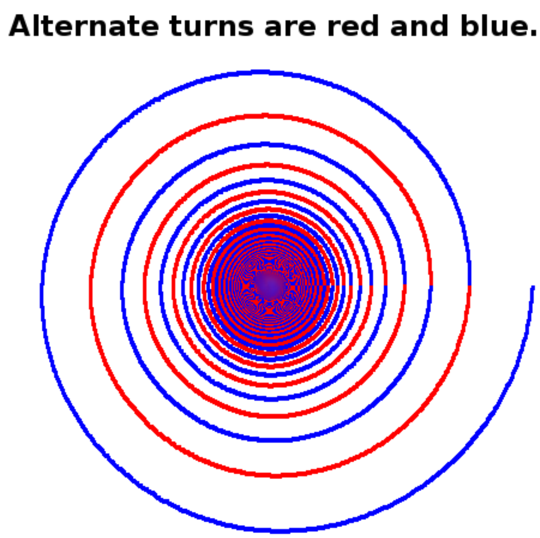
Exercise 2.28 *Suppose that a ball is dropped three meters and bounces half as high as the last fall each time. Find the vertical distance traveled by the ball.*

Exercise 2.29 *Suppose that a ball is dropped one meter and bounces 90% as high as the last fall each time. Find the vertical distance traveled by the ball.*

Exercise 2.30 *Suppose we have a series the members of which are computed as follows. Start with 10. To get the next member of the series, divide by two and add 3. Using geometric series, find the limit of the series. Hint: write out the first five terms of the series* without *simplifying.*

Exercise 2.31 *True or false: if we were to start the series in Problem 2.30 at 12 instead of 10 we would get the same answer. Explain why your answer is correct.*

Exercise 2.32 *Consider two jobs. The first has a \$10,000 starting salary and a raise of 5% each year. The second has a \$15,000 starting salary and a raise of 2% each year. How many years before the total earning of the first job are bigger than the total earning of the second job? Hint: algebra alone is not enough.*



Exercise 2.33 *The outermost turn of a spiral shown above is 10 meters long. Each turn of the spiral is 87.5% as long as the one before. Find the total length of the spiral to the nearest centimeter.*

Exercise 2.34 *Compute and simplify*

$$\sum_{n=0}^{\infty} \left(\frac{1}{x^2+1}\right)^n$$

given that x is a real number.

Exercise 2.35 *Suppose we have a string and snip out the middle quarter of the string, leaving us with two pieces. We then snip out the middle quarter of the remaining pieces, leaving us with four pieces of string. If we could do this forever, how much string would be left?*

Exercise 2.36 *Two men starting 20km apart each ride their bicycles directly toward one another at 10km/hr. A well trained bird flying at 50km starts at one bicyclist and flied to the other. As it reaches a cyclist it turns and flies back to the other. Assuming it can turn on a dime and instantly reverse it speed compute the distance traveled by the bird.*

2.3 Applications to Finance

Compound interest is fairly simple if we just let the money or debt accumulate. If, on the other hand, we make deposits or payments then different amounts of money are subject to interest for different amounts of time. This is a natural way in which we find series in finance.

Example 2.15 *Suppose that we deposit \$100.00 at the end of each month in a Christmas account, starting in January, and that the bank pays 5% annual interest compounded monthly. Assuming you draw the money out December 1st to do your gift shopping, what is the total amount and what was the interest? One thing we can do is figure the whole thing out, month by month:*

Month	*Balance*	*Interest*	*Deposit*
January	*0*	*0*	*100*
February	*100*	*0.42*	*100*
March	*200.42*	*0.84*	*100*
April	*301.26*	*1.26*	*100*
May	*402.52*	*1.68*	*100*
June	*504.20*	*2.10*	*100*
July	*606.30*	*2.53*	*100*
August	*708.83*	*2.95*	*100*
September	*811.78*	*3.38*	*100*
October	*915.16*	*3.81*	*100*
November	*1018.97*	*4.25*	*100*
December		**1123.33**	

Which means we have earned \$23.33 in interest over the course of eleven months. Lets look at this as a series problem. The monthly interest is $\frac{5}{12}\%$ which means that the multiplier is $1 + \frac{5}{1200} = \frac{1205}{1200} = \frac{241}{240}$ in lowest terms. This means the steps in the above table are:

$$\left(\cdots\left(\left(\left(100 \cdot \frac{241}{240}\right) + 100\right) \cdot \frac{241}{240} + 100\right) \cdot \frac{241}{240} \cdots + 100\right)$$

which is

$$100\left(\frac{241}{240}\right)^{10} + 100\left(\frac{241}{240}\right)^{9} + \cdots 100\left(\frac{241}{240}\right)^{8} + \cdots 100\left(\frac{241}{240}\right)^{2} + 100\left(\frac{241}{240}\right)^{1} + 100\left(\frac{241}{240}\right)^{0}$$

a finite geometric series with $d = 100$, $r = \frac{241}{240}$, and index n running from 0 to 10. Applying the geometric series formula we get

$$100 \cdot \frac{1 - \frac{241}{240}^{11}}{1 - \frac{241}{240}} \cong \$1123.21$$

The difference twelve cents is the result of more rounding the last cent up than down when the table was computed.

With only eleven compounding periods, the table in Example 2.15 is manageable. If we has an account where we were saving for a car for five years, or if our Christmas club account was compounded monthly, the size of the table would get annoying, rapidly. While we did a specific example in Problem 2.15, there is a general method lurking under the surface.

Fact 2.8 Formula for compound interest with deposits.
Suppose that a deposit of D is made for c compounding periods with a per-compounding period interest of $i\%$. Then, following the reasoning in Example 2.15 we get a final balance B of

$$B = D \cdot \frac{1-(1+\frac{i}{100})^c}{1-(1+\frac{i}{100})} = D \cdot \frac{(1+\frac{i}{100})^c - 1}{\frac{i}{100}}$$

Notice that the formula assumes that the first deposit is made at the end of the compounding period *after* interest is paid for that period. If the first deposit was made at the beginning of the compounding period we would get a slightly different formula, which you are asked to derive in the exercises.

Compounded how often? Continuously compounded interest.

Suppose we have an amount deposited at 5% annual interest for four years. How much does it matter how often the account is compounded?

Compounding period	Number of periods	Interest per period	Final Multiplier
Yearly	4	5%	$(1.05)^4 \cong 1.2155$
Monthly	48	5/12%	$(1.00417)^{48} \cong 1.2209$
Weekly	208	5/52%	$(1.0009615)^{208} \cong 1.2213$

Notice that the advantage of more compounding periods is not too large - just over quadrupling the number of compounding periods from 48 to 208 bought us an added multiplier of 0.0005 or fifty cents if we had \$1000.00 as our initial deposit.

It turns out that the effect of having more compounding periods does increase the multiplier but that increase is bounded above. That upper bound can be found by taking the limit as the number of compounding periods goes to infinity. Suppose we have an annual interest rate of r and c compounding periods. Then the multiplier for one year is

$$\left(1+\frac{r}{c}\right)^c$$

because we multiply by one plus the interest per compounding period a number of times equal to the number of compounding periods. The limit, to find the upper bound on the multiplier is:

$$\underset{c \to \infty}{Lim} \left(1+\frac{r}{c}\right)^c = e^r$$

The mechanics of computing this limit are beyond the scope of this course, but we can use the result which is called **continuously compounded interest**. It is also interesting to notice that e, the base of the natural logarithm, appears naturally in the process.

If principal of P is continuously compounded at a rate of r for t years then the final amount is

$$Pe^{rt}$$

Notice that t may be a decimal number of years.

Another similar problem to compound interest with deposits is paying off a loan in some number of equal payments.

Example 2.16 Amount of a loan payment.

Suppose we borrow \$3000.00 to pay for a used car at 6% annual interest, compounded monthly. If we want to pay the loan off in equal monthly payments over three years, what is the monthly payment?

Since we don't know the monthly payment amount we will make it a variable: p. We start with principal of \$3000.00 and at the end of each month we pay the interest and subtract a payment of p. The monthly interest is $\frac{6}{12} = \frac{1}{2}\%$ *so the multiplier in each compounding period is* (1.005). *This means that the amount of money goes:*

$$3000.00$$

$$3000.00 \cdot 1.005 - p$$

$$(3000.00 \cdot 1.005 - p) \cdot 1.005 - p$$

$$((3000.00 \cdot 1.005 - p) \cdot 1.005 - p) \cdot 1.005 - p$$

...and so on

So after three years (36 compounding periods) the fact the loan is paid off tells us that:

$$3000.00(1.005)^{36} - p(1.005)^{35} - p(1.005)^{34} - \cdots - p(1.005)^2 - p(1.005) - p = 0$$

And we see a geometric series. this means that

$$3000.00(1.005)^{36} = \sum_{n=0}^{35} p \cdot (1.005)^n$$

Simplifying both sides and applying the geometric series formula we get

$$3590.04 = p \cdot \frac{1.005^{36} - 1}{1.005 - 1}$$

$$3590.04 = 39.34p$$

$$p = \$91.26$$

The total amount paid is $\$91.26 \times 36 = \$3285.36$ *so we see that \$285.36 is the total interest paid over the course of the loan.*

As before, there is a general formula lurking in the specific example. If the geometric series caused by making payments is equal to the principal times a power of the multiplier equal to the number of compounding payments then we can get the payment.

Fact 2.9 Formula for the amount of a loan payment.
Suppose that the principal is P, the multiplier in each compounding period is m, and the number of compounding periods is c. Then, following example 2.16 we see that if p is the payment then

$$Pm^c = p \cdot \frac{m^c - 1}{m - 1}$$

Solving for p we get

$$p = \frac{Pm^c(m-1)}{m^c - 1}$$

Notice that this formula is simpler because it is done in terms of the multiplier m rather than the interest rate. Remember that if we are paying $i\%$ interest in a compounding period the multiplier is $m = 1 + \frac{i}{100}$.

We now turn to a problem quite similar to the last one. Given a monthly payment amount, an interest rate, and the amount of the original loan, we will find the number of payments that will retire a loan. Typically the last payment will be partial - we are actually looking for a number of months and so it is likely the last payment will overshoot the loan amount a bit.

Fact 2.10 Finding the number of payments of a given size. *Suppose we have a loan of \$L and pay \$p dollars a month with an annual interest rate of r%, compounded monthly. Borrowing steps from Example 2.16 and sliding to get everything with a c in it on the same side, we see that*

$$\frac{m^c}{m^c - 1} = \frac{p}{P(m-1)}$$

Recall that $m = 1 + \frac{r}{12 \times 100}$ *in this case. We now solve for c and get*

$$\begin{aligned}
\frac{m^c}{m^c - 1} &= \frac{p}{P(m-1)} \\
m^c &= (m^c - 1)\frac{p}{P(m-1)} \\
m^c &= m^c\frac{p}{P(m-1)} - \frac{p}{P(m-1)} \\
m^c - m^c\frac{p}{P(m-1)} &= -\frac{p}{P(m-1)} \\
m^c\left(1 - \frac{p}{P(m-1)}\right) &= -\frac{p}{P(m-1)} \\
m^c\left(\frac{p}{P(m-1)} - 1\right) &= \frac{p}{P(m-1)} \\
m^c &= \frac{\frac{p}{P(m-1)}}{\frac{p}{P(m-1)} - 1} \\
m^c &= \frac{p}{p - P(m-1)} \\
c &= log_m\left(\frac{p}{p - P(m-1)}\right)
\end{aligned}$$

Giving us a formula for the number of compounding periods c. Remember to round up to the nearest integer when using this formula.

Example 2.17 Problem: *Suppose we take a a loan for \$1000.000 at 5% interest compounded annually. If we pay \$200.00 at the end of each year, how many years will it take to pay off the loan?*

Strategy: Find the multiplier m and apply the formula for the number of payments. The multiplier is $1+\frac{5}{100} = 1.05$ *so we get*

$$log_{1.05}\left(\frac{200}{200 - 1000(1.05-1)}\right) = log_{1.05}4/3 = \frac{log(4/3)}{log(1.05)} \cong 5.89$$

rounding up, we see it will take six years to pay of the loan with a partial payment in the sixth year.

We've already seen that some numbers, negative ones, don't have logarithms. If we look at the formula for the number of payments c in Formula 2.10 then the formula will try to take the log of a negative number if $p - P(m-1)$ is negative. This is because if we don't have $p > P(m-1)$ then the payment is actually smaller than the interest that accrues and the amount that is owed grows larger in each compounding period.

Fact 2.11 Minimum amount of a payment
If we make payments p on a loan of P dollars with multiplier m then it must be true that $p > P(m-1)$ for the loan to ever be repaid. Equivalently, if i% is the interest rate in each period then we must have $p > P\frac{i}{100}$ for the loan to ever be paid off.

Exercises

Exercise 2.37 *Suppose a business makes quarterly deposits of \$50,000 to an account that pays 4% annual interest compounded quarterly. Make a table, like the one in Example 2.15 for three years (twelve quarters).*

Exercise 2.38 *Use a geometric series to find the final balance for Exercise 2.37. Compare the results - what is the monetary difference due to rounding?*

Exercise 2.39 *Use a geometric series to find the final balance for Exercise 2.40. Compare the results - what is the monetary difference due to rounding?*

Exercise 2.40 *Suppose a prudent student makes monthly deposits \$75.00 to an account that pays 3% annual interest compounded monthly. Make a table, like the one in Example 2.15 for two years (twenty-four months).*

Exercise 2.41 *Find the final account balance for each of the following monthly deposit amounts d, assuming interest is compounded monthly.*

a) Amount $d = \$50$ for five years at 3%. **b)** Amount $d = \$50$ for five years at 5%.

c) Amount $d = \$100$ for ten years at 2%. **d)** Amount $d = \$75$ for six years at 4/25%

e) Amount $d = \$50$ for twenty years at 4% **f)** Amount $d = \$10$ for fifty years at 3%

Exercise 2.42 *For each part of Problem 2.41 find the total interest earned (the amount above the total of the deposits).*

Exercise 2.43 *Find the monthly payment to retire each of the following loans with amount L at the stated rate of interest. Assume the interest is compounded monthly.*

a) Loan $L = \$5000$ for five years at 4%. **b)** Loan $L = \$5000$ for five years at 5%.

c) Loan $L = \$220,000$ for twenty years at 4\$. **d)** Loan $L = \$220,000$ for twenty years at 5\$.

e) Loan $L = \$200$ for two years at 15%. **f)** Loan $L = \$200$ for two years at 10%.

Exercise 2.44 *For each part of Problem 2.41 find the total interest paid (the amount above the original amount of the loan).*

Exercise 2.45 *For each of the following loan amounts L and monthly payment amounts p find the number of payments needed to retire the loan at the given interest rate. Assume that the interest is annual but that it is compounded monthly.*

a) *Loan* $L = \$5000$ *payment* $p = \$400$ *at 4%.* **b)** *Loan* $L = \$5000$ *payment* $p = \$400$ *at 5%.*

c) *Loan* $L = \$1,000,000$ *payment* $p = \$2600$ *at 3%.* **d)** *Loan* $L = \$1,000,000$ *payment* $p = \$3000$ *at 3%.*

e) *Loan* $L = \$2000$ *payment* $p = \$10$ *at 4%.* **f)** *Loan* $L = \$2000$ *payment* $p = \$5$ *at 4%.*

Exercise 2.46 *A loan, compounded monthly at an annual rate of 6%, is exactly paid off with 127 payments of \$100.00 each. What was the original amount of the loan?*

Exercise 2.47 *A loan, compounded monthly at an annual rate of 4%, is exactly paid off with 217 payments of $40.00 each. What was the original amount of the loan?*

Exercise 2.48 *You charge $1200 dollars on a card that charges 1% interest per month. If you forget about it for two years, what is you card's balance?*

Exercise 2.49 *You charge $1200 dollars on a card that charges 1% interest per month. What equal monthly payments will retire the debt in two years?*

Exercise 2.50 *You charge $1200 dollars on a card that charges 1% interest per month. What is the minimal payment to keep the debt from growing?*

Exercise 2.51 *You charge $1200 dollars on a card that charges 1% interest per month. If you forget about it for two years then what equal monthly payments will retire the debt by the end of the third year (in one more year)?*

Exercise 2.52 *A bank will give you a five year loan at $200,000 for a house at 5% interest or at 4.75% interest if you pay a processing fee of $1000.00 up front. The interest is annual, compounded monthly. Which is the better deal by the end of the loan? Decide by comparing the amount left to pay after five years.*

Exercise 2.53 *A bank will give you a three year loan at $100,000 for a house at 3.75% interest or at 3.25% interest if you pay a processing fee of $1200.00 up front. The interest is annual, compounded monthly. Which is the better deal by the end of the loan? Decide by comparing the amount left to pay after three years.*

Exercise 2.54 *If you deposit $200.00 per month at 5% annual interest, compounded monthly then for how many months do you have to make deposits to be able to withdraw $1000.00 a month forever, once you return and switch to making withdrawals?*

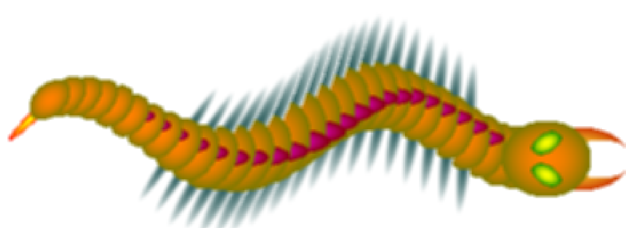
Exercise 2.55 *If you deposit $250.00 per month at 5% annual interest, compounded monthly then for how many months do you have to make deposits to be able to withdraw $1000.00 a month forever, once you retire and switch to making withdrawals?*

Exercise 2.56 *Suppose we have a $2000 loan compounded monthly at 4% annual interest. Make a table showing the number of payment periods required to retire the loan for each of the following monthly payment amounts: $5, $10, $15, $20, $25, $30, $40, and $50.*

Exercise 2.57 *Formula 2.8 assumes that the first deposit is made at the end of the first compounding period and so earns no interest for that compounding period. Find what the formula would be if the first deposit was made at the beginning of the first compounding period.*

Exercise 2.58 *Suppose that a company that sells leather couches offers 3% interest on the cost of a $800.00 couch with $50.00 monthly payments. Compare the terms "no payment for the first year", "no payment and no interest for the first year", and "start paying the first month" by computing the number of payments needed to pay off the couch. The comparison may turn on the decimal part of the number of payments.*

Exercise 2.59 *A payday loan shop charges 10% monthly interest, compounded daily. If you borrow $200.00 against a paycheck which will show up in 14 days, what is your repayment amount on the loan?*



Going Strong?

Chapter 3

Introduction to Derivatives

In this chapter we will be understand what a derivative is, how to tell if it exists or not, and how to find it if it does. The ideas of limits and continuity of a function are pillars on which we build the derivative so we begin with those concepts.

> *Calculus required continuity, and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be.*
> **– Bertrand Russell**

> *Yet in another way, calculus is fundamentally naive, almost childish in its optimism. Experience teaches us that change can be sudden, discontinuous, and wrenching. Calculus draws its power by refusing to see that. It insists on a world without accidents, where one thing leads logically to another. Give me the initial conditions and the law of motion, and with calculus I can predict the future – or better yet, reconstruct the past. I wish I could do that now.*
> **–Steven Strogatz**

> *Calculus has its limits.*
> **– Anon.**

3.1 Limits and Continuity

In the previous chapter we saw a bit about limits and how they are used to deal with sequences and series. This section will give a more in-depth treatment of how they are related to functions. Let's start with the notation for the limit of a function at a point.

Definition 3.1 *We write*

$$\lim_{x \to a} f(x) = L$$

for the limit of the function $f(x)$ as x approaches a. The notation also says that the value of the limit is L.

In plain English, $f(x)$ is the function we are dealing with, we have some number on the x axis a, and we want to know what is going to happen when we plug numbers that are *close* to a, but not equal to a, into $f(x)$. Depending on how close we get to a, that will determine how close the value of $f(a)$ is going to be to L. Here is the punchline, if we keep plugging in numbers that are getting closer and closer to a and those make $f(a)$ get closer and closer to L, then we say L is the limit of $f(x)$ as x approaches a. If $f(a)$ fails to get closer to L then L is not the limit. Let's do an example to make clear what we mean by "closer".

Example 3.1 Example of the limit of a continuous function.

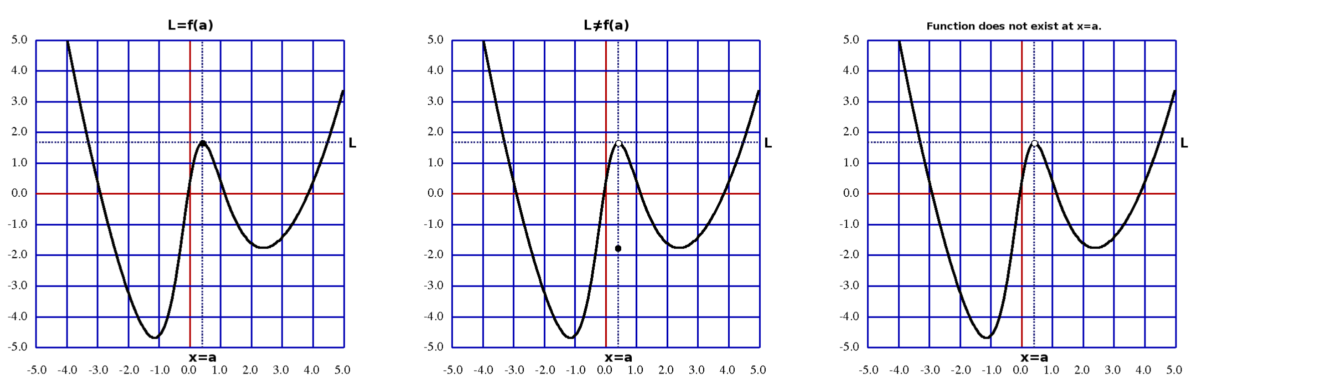
Consider the parabola $f(x) = x^2$. If we were to ask, what is the limit of x^2 as x approaches 3, we could use a table like the following to help us answer our question.

x	*f(x)*	*x*	*f(x)*
2.0	*4*	*4.0*	*16*
2.5	*6.25*	*3.5*	*12.25*
2.8	*7.84*	*3.2*	*10.24*
2.9	*8.41*	*3.1*	*9.61*
2.95	*8.7025*	*3.05*	*9.3025*
2.99	*8.9401*	*3.01*	*9.0601*
2.999	*8.994001*	*3.001*	*9.006001*

Clearly, as the numbers we plug in are getting closer and closer to 3, their values in the function are getting closer to 9. This means that 9 is the limit and we say that the limit of x^2 as x approaches 3 is 9. Using mathematical notation:

$$\lim_{x \to 3} x^2 = 9$$

At this point, you are probably asking why we didn't just plug 3 into x^2 and call it a day. Sometimes that will work, and when it does it is exactly the right way to get the limit. Not all of the functions we are going to be working with are as well behaved as x^2. They often require manipulation, have values that don't seem like they belong, or simply don't have values that are defined at certain points. Consider the following three graphs, hopefully they will illustrate many of the possibilities:



(a) (b) (c)

In part (a) $f(a) = L$, as we would normally expect. In part (b) $f(a) \neq L$ and in part (c) $f(a)$ is undefined. In every case, though,

$$lim_{x \to a} f(x) = L$$

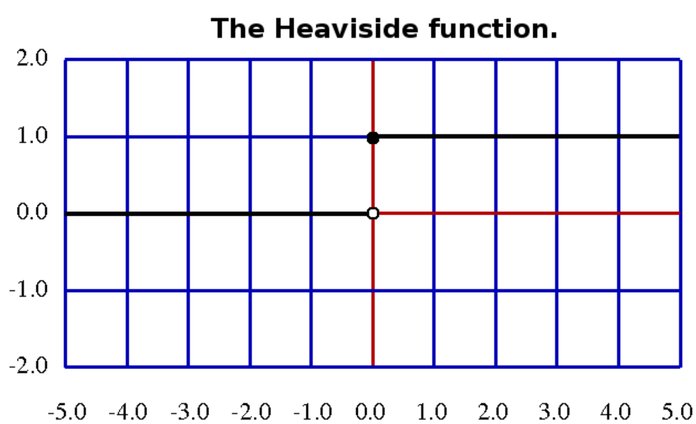
Notice that there is no requirement that the limit agree with the function - as we will see, however, having the function and the limit agree is a desirable state of affairs called *continuity*.

At this point we remind the reader of a convention used in the pictures above. A solid dot represents a point that is present. A hollow dot represents a point that is absent. The middle graph uses this convention to show that a point is not where the limit might cause us to expect it to be. This convention is also used in Example 3.2 to show the value of the Heaviside function at $x = 0$.

3.1.1 One-Sided Limits and Existence

In our earlier example, when we found the limit of x^2 in Example 3.1, we approached the number from both sides; from below $x = 3$ and from above $x = 3$. Sometimes we only need to find out what the limit of a function at a point is from one side, and by convention we say either the left or the right side. The notion of left and right follow from the standard presentation of graphs.

Example 3.2 *Consider the Heaviside function, $H(x)$:*



As a piecewise function, which is just another way of saying "this function is made up of different pieces", we write the Heaviside function as:

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

As you can see, the way we read this is: "If x is bigger than or equal to zero, then the function is equal to 1. If x is strictly less than zero, then the function is equal to zero." Let's take the limit of the Heaviside function as x approaches 0 from the left side. This means that we are plugging in numbers that are less than 0, getting closer and closer but never actually equalling 0. The result of this being:

x	*H(x)*
-1	*0*
-0.5	*0*
-0.25	*0*
-0.1	*0*
-0.01	*0*
-0.001	*0*
-0.0001	*0*

So we can see that the limit as x approaches 0 from the left is 0, and we write one-sided limits from the left as,

$$\lim_{x \to a^-} f(x) = L$$

or more specifically in the case of the Heaviside function as it approaches 0 from the left,

$$\lim_{x \to 0^-} H(x) = 0$$

We use a "-" to denote that this is a left-sided limit. If we were to use a "+", that would mean the limit is a right-sided limit, and we would write that as,

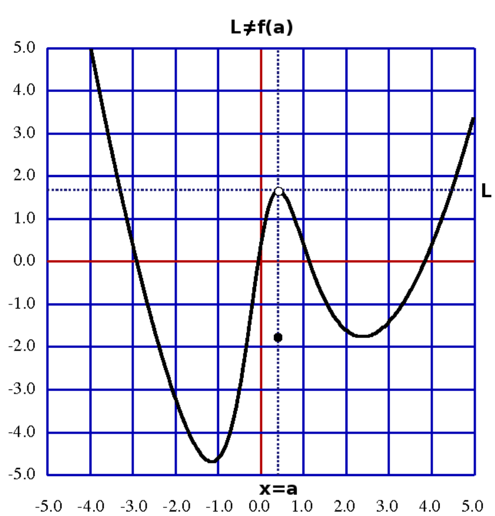
$$\lim_{x \to a^+} f(x) = L$$

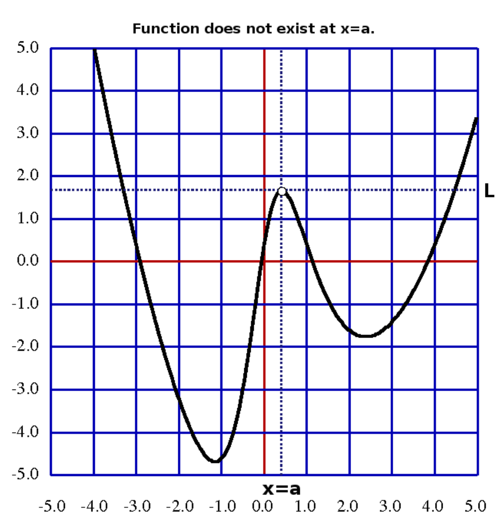
and if we were to take the right-sided limit of $H(x)$*, then we would write it as,*

$$\lim_{x \to a^+} H(x) = 1$$

One purpose of one-sided limits are to figure out whether a limit actually exists or not. Here is the key item to remember: if the left-sided limit equals the right-sided limit, then the limit exists. If the left-sided limit does not equal the right-sided limit, then the limit does not exist. Let's go over a couple of examples to drive this home.

Example 3.3 *If we recall 2 pictures we saw from earlier,*





In both of these pictures, the function does not equal the limit when $x = a$. It is important to remember that this fact has nothing to do with the limit. The limit in both cases is L. In the both pictures, we can see that as we move towards a from both the left and the right sides, we achieve the same result. Thus in both cases, the limit exists.

Example 3.4 *Returning to the Heaviside function, we already determined that the limit as x approached 0 from the left was equal to 0, and the limit as x approached 0 from the right was equal to 1. Thus,*

$$\lim_{x \to 0^-} H(x) \neq \lim_{x \to 0^+} H(x)$$

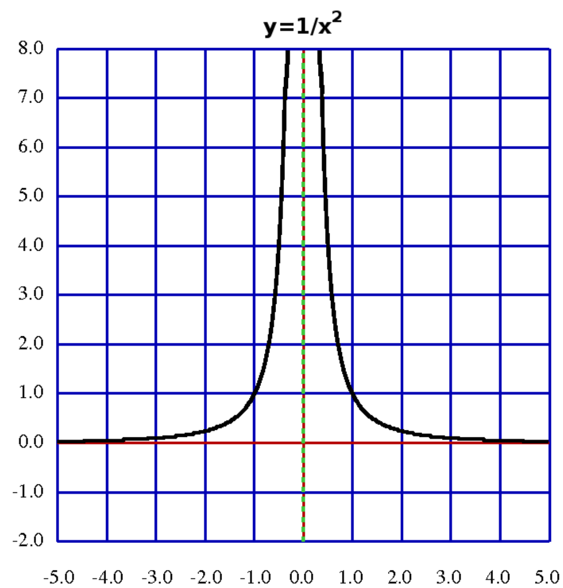
So the limit as x approaches 0 of $H(x)$ does not exist. Hopefully, this also makes clear the reason why in some cases we cannot simply plug in the value that the limit is approaching. If we did that in this case, then we would get $\lim_{x \to 0} H(x) = H(0) = 1$, which would be false.

To be clear: the limit of a function *completely* depends on the choice of a. Thus, if we were to take the limit of $H(x)$ as x approaches 2, the limits from the left and right would both be 1, and therefore the limit would exist.

So far we've covered the limit being a number, or not existing at all. There is a third case, in which the value grows (or shrinks) without bound. In this case, we use the symbol ∞, called *infinity*. Infinity is not a number, but a symbol used to show that something is forever increasing (or decreasing, in which case the $-\infty$ symbol is used).

Example 3.5 *Consider the function, $f(x) = \frac{1}{x^2}$, and its limit as x approaches zero. As before, we can't plug in zero to see what the answer is, since $\frac{1}{0}$ is undefined. We can use a table and a picture to see what is happening:*

x	$\frac{1}{x^2}$
± 1	*1*
± 0.5	*4*
± 0.2	*25*
± 0.1	*100*
± 0.01	*10000*
± 0.001	*1000000*



As x gets closer and closer to 0, the values of $\frac{1}{x^2}$ continue to grow larger without bound. Even though the limit from the left is the same as the limit from the right, ∞, the limit doesn't exist since the values of $\frac{1}{x^2}$ never get closer to any one number.

Fact 3.1 *The limit of a function $f(x)$ at a value $x = a$ exists and is equal to L if and only if the limits from the left and right both exist and are both also equal to L.*

3.1.2 Algebraic Properties of Limits

Assume c is a constant, $f(x)$ and $g(x)$ are functions, and the limits $lim_{x\to a}f(x) = L$ and $lim_{x\to a}g(x) = M$ exist. Here are some limit laws that will be useful when evaluating limits:

1. This is the **sum law**, which states that the limit of the sum is the sum of the limits.
$$\lim_{x\to a}[f(x) + g(x)] = L + M$$
.

2. This is the **difference law**, the limit of the difference is the difference of the limits.
$$\lim_{x\to a}[f(x) - g(x)] = L - M$$
.

3. The **scaling law**, the limit of a constant times a function is the same as the constant times the limit of a function.
$$\lim_{x\to a} c \cdot f(x) = c \cdot L$$
.

4. The **product law**, the limit of products is equal to the product of limits.
$$\lim_{x\to a} f(x) \cdot g(x) = L \cdot M$$
.

5. The **quotient law**, which say the limit of a quotient is equal to the quotient of the limits, given that the bottom limit is not zero. If $M \neq 0$ then
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}$$

There are two more details that might be obvious but need to be made explicit, even though they don't quite qualify being laws when it comes to limits:

1. $\lim_{x\to a} c = c$
2. $lim_{x\to a}x = a$

These can be read as taking the limit of a constant, no matter what number we approach on the x-axis, will just be that constant. The second item simply states that if x is the only item in the function you are taking the limit of, then a direct substitution will give you the limit.

Example 3.6 *Let $f(x) = 2x + 6$ and $g(x) = x^2 + 1$. We can use the Limit Laws to evaluate the following limits, if they exist.*

a) $\lim_{x\to 2}(f(x) + g(x))$ **b)** $\lim_{x\to -3} 2 \cdot f(x)$ **c)** $\lim_{x\to -1}\frac{f(x)}{2\cdot g(x)}$

Solutions

a) *We can use the sum law to separate the limits, and both the limits of $f(x)$ and $g(x)$ exist and can be evaluated without using tables (more on that later). So we get:*

$$\lim_{x\to 2}(2x + 6) + (x^2 + 1) = \lim_{x\to 2} 2x + \lim_{x\to 2} 6 + \lim_{x\to 2} x^2 + \lim_{x\to 2} 1 = 4 + 6 + 4 + 1 = 15$$

b) *Here we can use the scaling law, and the sum law:*

$$\lim_{x\to -3} 2\cdot(2x+6) = 2\cdot \lim_{x\to -3}(2x+6) = \lim_{x\to -3} 2x + \lim_{x\to -3} 6 = 2\cdot(-6+6) = 0$$

c) *In this case, we have to use the quotient law, the scaling law, and the sum law:*

$$\lim_{x\to -1} \frac{2x+7}{2\cdot(x^2+1)} = \frac{1}{2}\cdot \lim_{x\to -1}\frac{2x+6}{x^2+1} = \frac{1}{2}\cdot\frac{\lim_{x\to -1} 2x+6}{\lim_{x\to -1} x^2+1} = \frac{1}{2}\cdot\frac{\lim_{x\to -1} 2x + \lim_{x\to -1} 6}{\lim_{x\to -1} x^2 + \lim_{x\to -1} 1} = \frac{1}{2}\cdot\frac{-2+6}{1+1} = 1$$

3.1.3 The Function Growth Hierarchy

Sometimes we will be taking limits of a function as x approaches $\pm\infty$. In other words, we want to know what a function will do if we keep plugging in larger (or smaller) numbers forever. In some cases, this also causes the function to continually grow larger (or smaller) and we could write it as

$$\lim_{x\to\infty} f(x) = \infty$$

. Here is a useful guide in determining how "fast" a given function will approach ∞:

$$log(x) << x^n << e^x$$

We read this as, exponential functions grow a lot faster than polynomial functions, which grow a lot faster than logarithmic functions. Written there is log base 10, but if it were replaced with any other base, e or 2 for example, and it would still be true. As well, even though we use e^x, we could just as easily have written 4^x, or 10^x. Let's define what polynomial functions are, and then we'll see why we should bother learning how fast different functions grow.

Definition 3.2 *A* **polynomial** *is a sum of real-number multiples of whole number powers of a variable (like x).*

Example 3.7 *A polynomial looks like $x^4 - 2x + 1$. It only has variables, constants and whole number exponents. A polynomial function looks like $f(x) = x^4 - 2x + 1$, and if we plug a number, lets say 2 into $f(x)$, then $f(2) = 2^4 - 2(2) + 1 = 13$.*

Definition 3.3 *The* **degree** *of a polynomial is the highest power of x that appears in the polynomial.*

Example 3.8 degrees of polynomials
The degree of x^2 is 2, the degree of $1 - 2x - x^3$ is 3, the degree of $(x-1)(x+2)(x-3)(x+4)$ is 4 because it will contain a multiple of x^4 after we multiply it out.

A nice property of polynomials is that you can find the limit of a polynomial function simply by plugging in the number that x is approaching. This is because polynomial functions are *continuous*, which we will go over in the next section. For now, just trust in the fact that taking the limits of polynomial functions is very easy.

Example 3.9 *Find*

$$\lim_{x\to 2} 2x^2 - 4x + 10$$

To find the limit, simply plug 2 into the polynomial function,

$$\lim_{x\to 2} 2x^2 - 4x + 10 = 2(2)^2 - 4(2) + 10 = 10$$

Definition 3.4 *A* **rational function** *is a function in the form of a fraction whose numerator and denominator are polynomials.*

Examples of rational functions are $\frac{2x}{x^2}$, $\frac{x+1}{2x-1}$, $\frac{2}{x}$ and $\frac{x^2+2x^2+5x-7}{27x^3+14x^2+12x+9}$.

Now we finally get to the rule of thumb when it comes to taking the limit of rational functions.

Fact 3.2 *When taking the limit of a function $f(x)$ that is the ratio of two other functions, there are three possibilities:*

$$\begin{matrix} Lim \\ x \to \infty \end{matrix} \quad f(x) = \begin{cases} \infty \\ c \text{ a real constant} \\ 0 \end{cases}$$

As you can see, either there is no limit, there is a limit and it is a number, or the limit is 0. The key fact is that if the top is growing faster we go to ∞, if the bottom is growing faster we go to 0, and if they have similar growth rates we have work left to do. Let's do some examples to illustrate each possibility.

Example 3.10 *Let's compute the following limits:*

a) $\lim_{x\to\infty} f(x) = \frac{x^2}{log(x)}$ **b)** $\lim_{x\to\infty} f(x) = \frac{log(x)}{e^x}$ **c)** $\lim_{x\to\infty} f(x) = \frac{x^3-7x^2+1}{4x^3+10x^2}$

Solutions

a) *If we check our speed of growth hierarchy, we can see that the top function, x^2 grows much faster than the bottom, $log(x)$. The following table will tell us what is happening when we plug in increasing values of x:*

x	*f(x)*
10	100
100	5000
10000	25000000

As we can see, the value of $f(x)$ is growing without bound, thus we would write $\lim_{x\to\infty} f(x) = \infty$

b) *In this function the bottom function, e^x grows a lot faster than $log(x)$, so the bottom number is getting big relative to the top number. Let's check the following table to see what happens:*

x	$\frac{log(x)}{e^x}$
10	0.0000454
100	0
1000	0

The values aren't actually zero, but they are so close that the average calculator doesn't have enough digits to display the answer. Thus, the limit of this rational fraction is 0. Generally speaking, whenever the bottom function grows faster than the top function, the limit is going to be 0. If the top grows faster than the bottom, the limit does not exist, and we write it as $\pm\infty$ to show that its growing forever without bound.

c) *In this case, the speed of growth hierarchy won't help us, since the highest power of x on the top is equal to highest power of x on the bottom. Here is how we take the limit of a rational function whose two parts are both polynomials, same degree or not. The first step is to divide everything on the top by the highest power of x on the top. Separately, divide everything on the bottom by the highest power of x on the bottom.*

$$\lim_{x\to\infty} \frac{x^3-7x^2+1}{4x^3+10x^2} = \lim_{x\to\infty} \frac{\frac{x^3}{x^3} - \frac{7x^2}{x^3} + \frac{1}{x^3}}{\frac{4x^3}{x^3} + \frac{10x^2}{x^3}} = \lim_{x\to\infty} \frac{1 - \frac{7}{x} + \frac{1}{x^3}}{4 + \frac{10}{x}}$$

Now we can use the Limit Laws and speed of growth hierarchy to take the limit of $f(x)$.

$$\frac{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{7}{x} + \lim_{x\to\infty} \frac{1}{x^3}}{\lim_{x\to\infty} 4 + \lim_{x\to\infty} \frac{10}{x}}$$

Taking the limit of each piece of the rational function individually, we see that,

$$\frac{1-0+0}{4+0} = \frac{1}{4}$$

So we can see the limit of this rational is $\frac{1}{4}$, a constant.

Fact 3.3 Rule of thumb for rational functions
Suppose that $p(x)$ and $q(x)$ are polynomials so that

$$f(x) = \frac{p(x)}{q(x)}$$

is a rational function. Then

$$\underset{x \to \infty}{Lim} f(x) = \begin{cases} 0 & \text{if } q(x) \text{ is higher degree} \\ c & \text{if } q(x) \text{ has the same degree as } p(x) \\ \infty & \text{if } p(x) \text{ is higher degree} \end{cases}$$

Where c is the constant you get when you divide the highest power terms of $p(x)$ and $q(x)$.

Example 3.11 Applying the rule of thumb

The rule of thumb permits us to immediately calculate the following three limits.

1. $\underset{x\to\infty}{Lim} \frac{x}{x^2+1} = 0,$
2. $\underset{x\to\infty}{Lim} \frac{2x^2}{1-3x+5x^2} = \frac{2}{5},$
3. $\underset{x\to\infty}{Lim} \frac{x^4}{5x^3+2} = \infty,$

3.1.4 Continuity

This section deals with the concept of continuous functions. Informally, a continuous function can be thought of in the following way. If we were to graph the function and draw it from end to end with a pencil (assuming we had infinite lead and time in some cases), we would never have to lift the pencil off the paper. A function is continuous at a point if we can run our pencil over the area around the point, and through it, without taking the pencil off the paper. This is a good way to remember how to tell if a function is continuous at a point or not. Another is the graph of a continuous function has no breaks, holes or jumps in it. If drawing the function isn't really practical, we can use the following tools.

Definition 3.5 *A function $f(x)$ is continuous at a number a if*

$$\lim_{x\to a} f(x) = f(a)$$

What this definition implies is that if a function is continuous at a point a, then three things must be true:

1. $f(a)$ is defined and exists
2. $lim_{x\to a} f(x)$ exists

3. $lim_{x \to a} f(x) = f(a)$

A function that isn't continuous but is still defined around a point a is called discontinuous at a, or is said to have a discontinuity at a. If we consider the Heaviside function again, we can see that it is discontinuous at 0. The following examples will help make clear how to spot whether a function is continuous at a point or not.

Example 3.12 *Are the following functions continuous?*

a) $f(x) = \frac{x^2+1}{x-1}$ **b)** $f(x) = \begin{cases} x^2, & x \neq 2 \\ 5, & x = 2 \end{cases}$ **c)** $f(x) = \begin{cases} x^2 & x < 2 \\ 2x & x \geq 2 \end{cases}$ **d)** $f(x) = \begin{cases} x^2 & x < 2 \\ 2x & x > 2 \end{cases}$

Solutions

a) *If we consider* $\lim_{x \to 1} f(x)$*, we can see that the limit goes to* ∞*, thus it doesn't exist. Since the limit doesn't exist, this function is discontinuous at 1.*

b) *Even though the limit of this function as x approaches 2 exists,* $f(2) = 5 \neq \lim_{x \to 2} f(x) = 4$*. Since the limit doesn't equal the function evaluated at 2, this function is discontinuous at 2. Notice that this example is entirely artificial - we jammed the five in at* $x = 2$ *by using piecewise function notation.*

c) *To see if the limit exists, we need to check the right and left-sided limits.*

$$\lim_{x \to 2^-} f(x) = 4 = \lim_{x \to 2^+} f(x)$$

The limit exists at 2, the function is defined at 2, and the limit equals the value of $f(2)$*. Thus, this function is continuous at 2.*

d) *It is important to pay close attention to whether a function is defined at a certain point. This function is almost the same as the function in* **c)***, except that it isn't defined at 2. If we were to graph this, there would be a hole in the graph at* $f(2)$*. This function has a discontinuity at 2.*

Definition 3.6 *A function is continuous* **on an interval [a,b]** *if it is continuous at every point in the interval. If a function is continuous everywhere we simply say it is continuous.*

The combination of continuous functions is also continuous, similar to the limit laws. If $f(x)$ and $g(x)$ are continuous functions, then the following are also continuous:

- $f(x) + g(x)$
- $f(x) - g(x)$
- $c \cdot f(x)$, where c is a constant
- $f(x) \cdot g(x)$
- $\frac{f(x)}{g(x)}$, as long as $g(x) \neq 0$

Fact 3.4 Continuity of polynomial and rational functions

a) *Polynomial functions are continuous everywhere.*
b) *Rational functions are continuous at every point where they are defined.*

This fact is the reason we can take the limits of polynomial and rational functions simply by substituting numbers in for x. In fact you can take the limit of many continuous function by just substituting in a number for the variable. Only discontinuous functions or functions that make sudden or abrupt changes of direction require formally taking limits. This latter sort of function uses split rules. As we will see, this is an issue when we start working with derivatives. Here are some more useful facts connected with continuity.

Fact 3.5 other continuous functions.
The functions $y = c^x$ and $y = log_c(x)$ are both continuous for any constant $c > 0$ at all points where they exist. The function $y = \sqrt[n]{x}$ is continuous where it exists.

Exercises

Exercise 3.1 *For each of the following functions, make a table like that in Example 3.1 and give your opinion, based on the table, if the function has a limit at the given value of x.*

a) $f(x) = x^3$ at $x = 1$. **b)** $f(x) = ln(x)$ at $x = 3$. **c)** $f(x) = \frac{x+2}{x-1}$ at $x = 2$.

d) $f(x) = \frac{x^2-4}{x^2-9}$ at $x = 3$. **e)** $f(x) = e^{-1/x^2}$ at $x = 0$. **f)** $f(x) = 4^{1/x}$ at $x = -1$.

Exercise 3.2 *Which of the following functions have limits at the given point? Remember to check that the limits from both sides exist and agree.*

a) $f(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0 \end{cases}$. **b)** $f(x) = \begin{cases} 2x & x < 0 \\ x^2 & x \geq 0 \end{cases}$. **c)** $f(x) = \begin{cases} x^2 & x < 1 \\ 2x - 1 & x \geq 1 \end{cases}$.

d) $f(x) = \begin{cases} x^2 + 2x + 1 & x < 0 \\ 2x - 1 & x \geq 0 \end{cases}$. **e)** $f(x) = \begin{cases} e^x & x < 0 \\ x + 1 & x \geq 0 \end{cases}$. **f)** $f(x) = \begin{cases} x^3 + 1 & x < 1 \\ 2x^2 - 2 & x \geq 1 \end{cases}$.

Exercise 3.3 *Suppose that $H(x)$ is the Heaviside function. Which of the following functions have limits at the given point? Remember to check that the limits from both sides exist and agree.*

a) $f(x) = H(x) + H(-x)$ at $x = 0$. **b)** $f(x) = H(x) \times H(x)$ at $x = 0$. **c)** $f(x) = H(2 - x)$ at $x = 2$.

d) $f(x) = H(2 - x)$ at $x = 1$. **e)** $f(x) = (2 \times H(x) - 1)^2$ at $x = 0$. **f)** $f(x) = x \times H(x)$ at $x = 0$.

Exercise 3.4 *Compute the following limits or give a brief explanation of why they cannot be computed.*

a) $\underset{x\to 1}{Lim}\, x^2 + 3x + 1$. **b)** $\underset{x\to 4}{Lim}\,(x + 2)(x + 3)$.

c) $\underset{x\to 3}{Lim}\, \frac{x^3+1}{x^2+1}$. **d)** $\underset{x\to 2}{Lim}\, \frac{x^4-4}{x^2+4}$.

e) $\underset{x\to 0}{Lim}\, \frac{e^x+1}{e^{-x}+1}$. **f)** $\underset{x\to 0}{Lim}\, \frac{ln(x+2)}{ln(x+3)}$.

g) $\underset{x\to -2}{Lim}\, \frac{x^2-4}{x^2+4}$. **h)** $\underset{x\to -3}{Lim}\, \frac{x^2+9}{x^2-9}$.

i) $\underset{x\to -4}{Lim}\, xe^x$. **j)** $\underset{x\to -1}{Lim}\, xe^{-x}$.

Exercise 3.5 *Compute the following limits.*

a) $\underset{x\to\infty}{Lim}\, \frac{1}{x^2}$. **b)** $\underset{x\to\infty}{Lim}\, \frac{1}{x^2+1}$.

c) $\underset{x\to\infty}{Lim}\, \frac{x^2+x+1}{1-3x+x^2}$. **d)** $\underset{x\to\infty}{Lim}\, \frac{(2x+1)(2x-1)}{(x-3)(x+4)}$.

e) $\underset{x\to\infty}{Lim}\, \frac{x^3-1}{x^2+1}$. **f)** $\underset{x\to\infty}{Lim}\, \frac{x(x+2)(x-2)}{(x-1)(x+1)}$.

Exercise 3.6 *Compute the following limits. Looking at the graph of the functions may be inspirational.*

a) $\underset{x\to 0}{Lim}\, \frac{1}{x^4}$. **b)** $\underset{x\to 0}{Lim}\, \frac{1}{x^3}$.

c) $\underset{x\to 1}{Lim}\frac{x^2}{x^4-1}$. **d)** $\underset{x\to -1}{Lim}\frac{x^2}{x^4-1}$.

e) $\underset{x\to -2}{Lim}\frac{x^2-5x+6}{x^2+5x+6}$. **f)** $\underset{x\to 2}{Lim}\frac{x^3}{x^4-16}$.

Exercise 3.7 *Suppose we have three functions, $f(x)$, $g(x)$, and $h(x)$ so that for some constant a we have that:*

$$\underset{x\to a}{Lim} f(x) = 3 \qquad \underset{x\to a}{Lim} g(x) = 0 \qquad \underset{x\to a}{Lim} h(x) = -2$$

Compute each of the following limits.

a) $\underset{x\to a}{Lim}\,(f(x)+h(x))$. **b)** $\underset{x\to a}{Lim}\,(2f(x)-3h(x))$. **c)** $\underset{x\to a}{Lim}\left(\frac{f(x)}{g(x)}\right)$.

d) $\underset{x\to a}{Lim}\left(\frac{h(x)}{f(x)-3}\right)$. **e)** $\underset{x\to a}{Lim}\,(f(x)-h(x))^2$. **f)** $\underset{x\to a}{Lim}\,(f(x)g(x)+f(x)h(x)+g(x)h(x))$.

Exercise 3.8 *Find any points at which any of the following functions are not continuous.*

a) $f(x)=\frac{x^2+1}{x^2-4}$. **b)** $f(x)=\frac{x^2-1}{x^2+4}$. **c)** $f(x)=\frac{e^x}{e^x-3}$.

d) $f(x)=\frac{e^x}{e^{2x}-9}$. **e)** $f(x)=log(x^2-4)$. **f)** $f(x)=\sqrt{x^2-9}$.

Exercise 3.9 *Consider the function $f(x)=\frac{x^2-4}{x-2}$. At $x=2$ the function has a value of $\frac{0}{0}$ which is undefined. Compute the limit of the function at $x=2$. Does this require a table or can it be done with algebra?*

Exercise 3.10 *Following Problem 3.9, find* $\underset{x\to 5}{Lim}\frac{x^2-25}{x-5}$.

Exercise 3.11 *Compute* $\underset{x\to\infty}{Lim}\frac{log(x^5)}{x+5}$.

Exercise 3.12 *Compute* $\underset{x\to\infty}{Lim}\frac{exp(2x-1)}{x-1}$.

Exercise 3.13 *Compute* $\underset{x\to\infty}{Lim}\frac{e^x+1}{ln(x)+1}$.

Exercise 3.14 *Compute* $\underset{x\to\infty}{Lim}\frac{e^x+e^{-x}}{e^x-e^{-x}}$.

Exercise 3.15 *Suppose that*

$$f(x)=\begin{cases} x^2 & x<a \\ 2x+3 & x\geq a\end{cases}$$

Find all values of a that make the function continuous.

Exercise 3.16 *Suppose that*

$$f(x)=\begin{cases} x^2 & x<a \\ x+2 & x\geq a\end{cases}$$

Find all values of a that make the function continuous.

Exercise 3.17 *Suppose that*

$$f(x)=\begin{cases} x^3 & x<a \\ 4x & x\geq a\end{cases}$$

Find all values of a that make the function continuous.

Exercise 3.18 *Suppose that*

$$f(x) = \begin{cases} x^3 & x < a \\ 6x^2 - 11x + 6 & x \geq a \end{cases}$$

Find all values of a *that make the function continuous.*

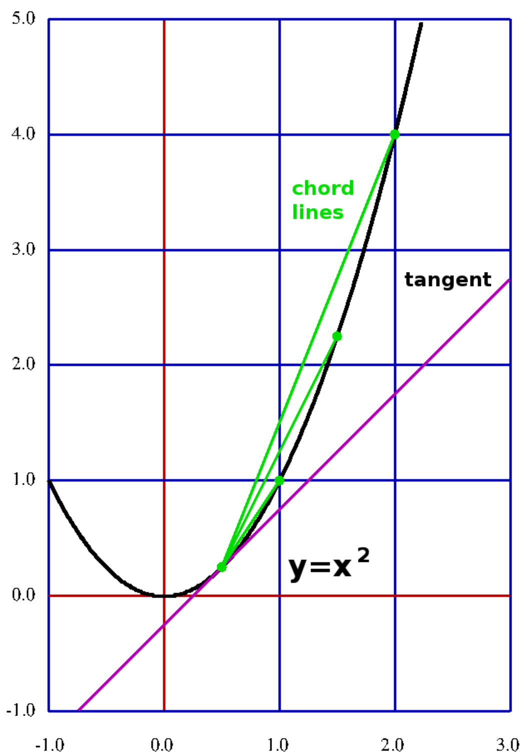
Exercise 3.19 *If* $H(x)$ *is the Heaviside function, demonstrate that* $(H(x-a))^2 = H(x-a)$ *for any constant* a*.*

Exercise 3.20 *Is there a difference in the value, which is not a number in any case, of the limit as* x *approaches zero for* $\frac{1}{x^2}$ *and* $\frac{1}{x^3}$*? Explain.*

Exercise 3.21 *Using the technique of dividing through the top and bottom of a rational function by the highest power of* x *that appears, demonstrate logically that the rule of thumb for rational functions is correct.*

3.2 Derivatives

Having waded through the sea of limits and continuity, we are now ready to harvest a useful tool.



Definition 3.7 *A* **tangent line** *is a line that touches a curve at a single point.*

Fact 3.6 *If a curve is measuring a quantity, like distance or money, at a given time, then the slope of the tangent line at a time* t *measures how fast the quantity is changing at time* t*.*

The entire point of this section is to develop the mathematical tools needed to find the slope of tangent lines to continuous curves. The reason this is hard is because two points define a line, but one point defines a tangent line. We are going to have to use a limit to extract the information to build the tangent line; information that would normally be in the second point. The starting point of the effort is to use chords of the curve.

Definition 3.8 *A* **chord** *of a curve is a line that joins two points on the curve.*

The picture at the left shows the tangent line to the curve $y = x^2$ at $x = 0.5$ in purple and several chords with $x = 0.5$, $y = 0.25$ at one end of the chord.

Example 3.13 Finding a tangent line*.*
Problem: *find the tangent line to* $y = x^2$ *at the point (1,1).*

Strategy: *If we fix one point of a collection of chords at (1,1) and move the other point toward (1,1) then the slopes of the chords will approach the chords of the tangent. The slope of chord between the point (1,1) and the point* $(1+h, (1+h)^2)$ *can be computed with the usual formula for the slope of a line between two points:*

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(1+h)^2 - 1}{1+h-1} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h$$

If the slope of the chords is $2+h$ then we can compute

$$\lim_{h \to 0} 2+h = 2$$

Now that we know that the slope of the tangent line to $y = x^2$ at $x = 1$ is $m = 2$ we can find the tangent line with the point-slope formula. The point is (1,1) so

$$(y-1) = 2(x-1)$$
$$y = 2x - 2 + 1$$
$$y = 2x - 1$$

The slope of the tangent line is a central object in calculus. The technique we used to extract the tangent line in the example can be generalized by the following definition.

Definition 3.9 *The* **derivative** *of a function $f(x)$ at a point $x = a$ is given by the limit*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The derivative at a is the slope of the tangent line to the graph of $f(x)$ at a.

Once we have a way of taking the derivative of a function at a point, we can take the derivative in general and get a formula for the derivative at any point.

Definition 3.10 *The* **derivative** *of a function $f(x)$ is given by the limit*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Notice that we use a "prime" to denote a derivative. If $f(x)$ is a function then its derivative is $f'(x)$, spoken $f - prime - of - x$.

Example 3.14 Computing a general derivative

Problem: *find the derivative of $f(x) = x^3$.*

Strategy: *Use the limit given in the definition.*

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2}{h} \\
&= \lim_{h \to 0} \frac{\not{h}(3x^2 + 3xh + h^2)}{\not{h}} \\
&= \lim_{h \to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2
\end{aligned}$$

So we have that if $f(x) = x^3$ then $f'(x) = 3x^2$.

Example 3.15 Computing a harder general derivative

Problem: *find the derivative of $f(x) = \frac{1}{x}$.*

Strategy: *Use the limit given in the definition.*

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{\frac{x}{x} \times \frac{1}{x+h} - \frac{(x+h)}{(x+h)} \times \frac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\
&= \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h} \\
&= \lim_{h \to 0} \frac{-h}{x(x+h)} \times \frac{1}{h} \\
&= \lim_{h \to 0} \frac{-\cancel{h}}{x(x+h)} \times \frac{1}{\cancel{h}} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x(x+0)} \\
&= \frac{-1}{x^2}
\end{aligned}$$

So we have that if $f(x) = \frac{1}{x}$ then $f'(x) = \frac{-1}{x^2}$.

3.2.1 Derivative Rules

Examples 3.14 and 3.15 show that, while knowing the derivative of a function is valuable, we *really* need a better way of computing derivatives. Notice that for both $f(x) = x^2$ and $g(x) = x^3$ all but one term in the expression we got from the definition of the derivative cancelled out. It turns out that this is true in general.

Fact 3.7 The power rule

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

The power rule tells us the derivative of any power of a variable. That's nice, but we need to extend the reach of our derivative techniques. The key point is this: because derivatives are based on limits, many of the rules for limits also hold for derivatives.

Fact 3.8 Linearity of Derivatives
If c is a constant and $f(x)$ and $g(x)$ are functions then:

1. $(c \times f(x))' = c \times f'(x)$
2. $(f(x) + g(x))' = f'(x) + g'(x)$
3. $(f(x) - g(x))' = f'(x) - g'(x)$

If we speak the power rule in English it is "multiply by the old power and subtract one from the power". With the rules above, we now have enough rules to take the derivative of any polynomial function without resorting to the derivative.

Example 3.16 Problem: *If $f(x) = x^2 + 3x + 5$ find $f'(x)$.*

Solution:

$$\begin{aligned} f'(x) &= \left(x^2 + 3x + 5\right)' \\ &= \left(x^2 + 3x^1 + 5x^0\right)' \\ &= \left(x^2\right)' + \left(3x^1\right)' + \left(5x^0\right)' \\ &= \left(x^2\right)' + 3\left(x^1\right)' + 5\left(x^0\right)' \\ &= 2 \times x^1 + 1 \times 3x^0 + 0 \times 5x^{-1} \\ &= 2x + 3 \times 1 + 0 \\ &= 2x + 3 \end{aligned}$$

The above example is done with a great deal of formality; normally a polynomial derivative can just be done informally. For example:

$$\left(x^3 + 5x^2 + 7\right)' = 3x^2 + 10x$$

One of the transformations in Example 3.16 is the somewhat artificial $5 = 5 \times x^0$ which relies on the fact that $x^0 = 1$. Recall that a derivative is the rate at which a quantity is changing. Since constants don't change, the following fact is obvious.

Fact 3.9 Derivative of a constant
The derivative of any constant is zero.

While we will not go through the logical demonstrations that the following three rules are correct, they can all be derived from the limit definition of derivative with a bit of algebraic skill.

Fact 3.10 Product, Quotient, and Reciprocal Rules

Suppose that $f(x)$ and $g(x)$ are functions that have derivatives. Then:

1. $(f(x) \times g(x))' = f'(x) \times g(x) + f(x) \times g'(x)$ **(the product rule)**
2. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ **(the quotient rule)**
3. $\left(\frac{1}{f(x)}\right)' = \frac{-f'(x)}{f(x)^2}$ **(the reciprocal rule)**

Example 3.17 *Let's practice the new rules. First the product rule:*

$$\begin{aligned}\left(x^4 \times (x^2+1)\right)' &= 4x^3 \times (x^2+1) + x^4 \times 2x \\ &= 4x^5 + 4x^3 + 2x^5 \\ &= 6x^5 + 4x^3\end{aligned}$$

Notice that, in this case, it would have been perfectly possible to multiply the expression out and then take the derivative: check that the same answer results. While either method is perfectly workable, you must decide which one is less work.

Next the quotient rule:

$$\begin{aligned}\left(\frac{x^3}{x^2+1}\right)' &= \frac{(x^2+1) \times 3x^2 - x^3 \times 2x}{(x^2+1)^2} \\ &= \frac{3x^4 + 3x^2 - 2x^4}{(x^2+1)^2} \\ &= \frac{x^4 + 3x^2}{(x^2+1)^2}\end{aligned}$$

It might seem like a good idea to multiply the bottom of the answer out to get $x^4 + 2x^2 + 1$. In practice this usually isn't a good idea. This is especially true if we must make further manipulations of the expression, as we will see in Chapter 4.

We finish up the example with a demonstration of the reciprocal rule.

$$\left(\frac{1}{x^3+1}\right)' = \frac{-3x^2}{(x^3+1)^2}$$

We've built up the mathematical muscle to deal with both polynomial and rational functions, but we have not yet touched on the exponential and logarithm functions.

Fact 3.11 derivatives of logs and exponentials

1. $(ln(x))' = \frac{1}{x}$
2. $(log_b(x))' = \frac{1}{ln(b) \times x}$
3. $(e^x)' = e^x$
4. $(c^x)' = ln(c) \times c^x$

Example 3.18 Problem: *Find the tangent line to $y = 2^x$ at $x = 1$.*

The point of tangency is (1,2) so all we need is the slope of the line. The derivative is the slope of the line so:

$$f'(x) = ln(2) \cdot 2^x$$

$$m = f'(a) = ln(2) \cdot 2^1 = 2ln(2)$$

Now that we have the slope we build the line with the point-slope form

$$\begin{aligned}
(y-2) &= 2ln(2)(x-1) \\
y &= 2ln(2)x - 2ln(2) + 2 \\
y &= 2ln(2)x + 2(1 - ln(2)) \\
y &\cong 1.386x + 0.614
\end{aligned}$$

Example 3.19 Problem: *Find the value of x so that $y = e^x$ has a tangent-slope of 1 and then find the tangent line.*

The tangent slope is the derivative so we must solve for the x that makes the derivative equal to 1.

$$(e^x)' = 1 \tag{3.1}$$
$$e^x = 1 \tag{3.2}$$
$$ln(e^x) = ln(1) \tag{3.3}$$
$$x = 0 \tag{3.4}$$
$$\tag{3.5}$$

This means that the point of tangency is $(0, e^0) = (0, 1)$. We started knowing that the slope was $m = 1$. So the tangent line is:

$$\begin{aligned}
(y-1) &= 1 \times (x-0) \\
y - 1 &= x \\
y &= x + 1
\end{aligned}$$

3.2.2 Functional Composition and the Chain Rule

We have one derivative rule left to learn. It is one of the most powerful and also one of the most confusing. Let's start with an example that motivates our need for the additional rule.

Example 3.20 Problem: *Suppose that $f(x) = x^5$ and that $g(x) = x^2 + 1$. Find the derivative of $f(g(x))$. Compute:*

$$\begin{aligned}
f(g(x))' &= \left[\left(x^2+1\right)^5\right]' \\
&= \left[x^{10} + 5x^8 + 10x^6 + 10x^4 + 5x^2 + 1\right]' \\
&= 10x^9 + 40x^7 + 60x^5 + 40x^3 + 10x
\end{aligned}$$

The second step above, multiplying out the fifth power, was done using a fact called the binomial theorem *which you can look up if you want to. Contemplate, however, how annoying it would be to multiply out without a trick. It turns out that the situation is even worse. Let's continue the computation (using the binomial theorem again).*

$$\begin{aligned} 10x^9 + 40x^7 + 60x^5 + 40x^3 + 10x &= 10x(x^8 + 4x^6 + 6x^4 + 4x^3 + 1) \\ &= 10x(x^2 + 1)^4 \end{aligned}$$

Which is easier to work with in a lot of cases. The other thing is that something like *a power rule happened. The power of $x^2 + 1$ went from 5 to 4, its just that some other stuff happened as well. What that other stuff is can be specified precisely by our last derivative rule.*

Definition 3.11 *If $f(x)$ and $g(x)$ are functions then the* **composition** *of f and g is the result of plugging $g(x)$ into $f(x)$ as if it were the variable. We denote the composition by $f(g(x))$ or by $(f \circ g)(x)$.*

As you would expect, the last of our derivative rules deals with functional composition.

Fact 3.12 The chain rule
If $f(x)$ and $g(x)$ are functions then

$$(f(g(x)))' = f'(g(x)) \times g'(x)$$

Example 3.21 Using the chain rule

1 *Suppose that $f(x) = x^5$ and that $g(x) = x^2 + 1$. Find the derivative of $f(g(x))$ using the chain rule. Notice that $f'(x) = 4x^5$ and $g'(x) = 2x$. Then if we apply the chain rule we get:*

$$(f(g(x)))' = 5\left(x^2 + 1\right)^4 \times 2x = 10x\left(x^2 + 1\right)^4$$

Which is the answer we got in the earlier example.

2 *If $h(x) = e^{x^2}$, find $h'(x)$.*

Let's rephrase $h(x)$ as a composition. Set $f(x) = e^x$ and $g(x) = x^2$. Then

$$f(g(x)) = e^{x^2} = h(x)$$

We then compute $f'(x) = e^x$ and $g'(x) = 2x$ and apply the chain rule to get

$$h'(x) = e^{x^2} \times 2x = 2xe^{x^2}$$

Exercises

Exercise 3.22 *Use the limit based definition of the derivative to compute the derivative of each of the following functions. The algebra on the last two functions can get a bit intense.*

a) $f(x) = x^2$. **b)** $f(x) = x^4$. **c)** $f(x) = \frac{1}{x^2}$.

d) $f(x) = \frac{1}{x^3}$. **e)** $f(x) = \frac{1}{x^2+1}$. **f)** $f(x) = \frac{x}{x^2+1}$.

Exercise 3.23 *For each of the following functions, find the derivative at the indicated value of x and use it to compute the tangent line for that value of x.*

a) $f(x) = x^2$, $x = 2$. **b)** $f(x) = x^3$, $x = 2$. **c)** $f(x) = \frac{1}{x}$, $x = 1$.

d) $f(x) = \frac{1}{x^2}$, $x = -1$. **e)** $f(x) = \frac{1}{x^2+1}$, $x = \frac{1}{2}$ **f)** $f(x) = \frac{x}{x^2+1}$, $x = 2$.

Exercise 3.24 *For each of the following functions, find the derivative using the product rule.*

a) $f(x) = (x^2 + x + 1)(x^2 - 3x - 2)$ **b)** $f(x) = (x^3 + 1)(x + 4 + 3x^2 + 1)$ **c)** $f(x) = xe^x$ **d)** $f(x) = x^2 e^{2x}$

e) $f(x) = x^2 ln(x)$ **f)** $f(x) = xln(x^2 + 1)$

Exercise 3.25 *For each of the following functions, find the derivative using the reciprocal rule.*

a) $f(x) = \frac{1}{x^4+1}$ **b)** $f(x) = \frac{1}{x^3+1}$ **c)** $f(x) = \frac{2}{xln(x)}$

d) $f(x) = \frac{2}{1+x+x^2+x^3}$ **e)** $f(x) = \frac{1}{e^x+e^{2x}}$ **f)** $f(x) = \frac{1}{ln(x+1)}$

Exercise 3.26 *For each of the following functions, find the derivative using the quotient rule.*

a) $f(x) = \frac{x}{x^2+1}$ **b)** $f(x) = \frac{x+1}{2x-1}$ **c)** $f(x) = \frac{x^2}{3x-2}$

d) $f(x) = \frac{x^3}{x^2-1}$ **e)** $f(x) = \frac{e^x}{e^x+2}$ **f)** $f(x) = \frac{e^x}{x^3+1}$

Exercise 3.27 *For each of the following functions, find the derivative using the chain rule. You may also need other rules.*

a) $f(x) = (x^2 + 1)^3$ **b)** $f(x) = (x^3 + 1)^5$ **c)** $f(x) = \sqrt{x^2 + x + 1}$ **d)** $f(x) = \sqrt{x^3 - x^2 + x - 1}$

e) $f(x) = e^{x^5}$ **f)** $f(x) = e^{x^3+3x+2}$ **g)** $f(x) = \frac{x^2}{(x^3+1)^5}$ **h)** $f(x) = \frac{(2x+1)^8}{(1-x)^5}$

i) $f(x) = (e^{2x} + 1)^5$ **j)** $f(x) = (ln(x^2 + 1) + 2)^6$

Exercise 3.28 *Find the tangent line at the indicated value of x.*

a) $f(x) = (x^2 + 1)^4$ at $x = 0$. **b)** $f(x) = (x^2 - 2x + 1)^3$ at $x = 0$. **c)** $f(x) = \frac{x+1}{x-1}$ at $x = 2$.

d) $f(x) = \frac{x}{x^2+1}$ at $x = 1$. **e)** $f(x) = \frac{1}{x} + \frac{1}{x+1}$ at $x = 1$. **f)** $f(x) = \frac{1}{x^2} + \frac{1}{(x-1)^2}$ at $x = -1$.

Exercise 3.29 *Find all tangent lines to* $f(x) = \frac{2}{x^2+1}$ *that are parallel to* $y = x$.

Exercise 3.30 *Find all tangent lines to* $f(x) = x^3 - 4x$ *that are parallel to* $y = x$.

Exercise 3.31 *Find all horizontal tangent lines to* $y = x^3 - 27x$.

Exercise 3.32 *Find all horizontal tangent lines to* $y = \frac{3}{x^2-4x+5}$.

Exercise 3.33 *Find the derivative of*

$$y = \frac{\left(Ln(x^2 + 2) + 1\right)^3}{Ln(x^2 + 2) - 3}$$

Exercise 3.34 *Find the derivative of*

$$y = \frac{\left(e^{2x} + 5\right)^3}{e^{2x} + 1}$$

Exercise 3.35 *Suppose the dollars of profit on making x units of an MP3 player is* $P(x) = \frac{2x^3+x^2+2x}{x^2+1}$. *Find the rate at which the profit is changing (the derivative of profit) and compute the marginal profit of manufacturing the first, tenth, and hundredth unit.*

Exercise 3.36 *Suppose the profit on making x car, in thousands of dollars, is* $P(x) = \frac{3.5x^4+8.1x^2}{x^3+3x+1}$. *Find the rate at which the profit is changing (the derivative of profit) and compute the marginal profit of manufacturing the fourth, twentieth, and fiftieth car.*

Exercise 3.37 *Suppose that $P(x)$ measures the profit, in dollars, of making x units of some device. If*

$$\underset{x \to \infty}{Lim} P'(x) = 2$$

what does this tell us about profit per unit? Ignore the logical problem that manufacturing an infinite number of units is impossible. We often use infinite limits as surrogates for large finite numbers.

Exercise 3.38 *Suppose that $P(x)$ measures the profit, in dollars, of making x units of some device. Demonstrate through argumentation that if the formula P is accurate for all x then*

$$\underset{x \to \infty}{Lim} P'(x) \leq 0$$

Exercise 3.39 *Demonstrate logically that the reciprocal rule is a special case of the quotient rule.*

Exercise 3.40 *Suppose that $f(x) = e^{g(x)}$ for a function $g(x)$ that has a derivative. Prove that $f(x)$ and $g(x)$ have horizontal tangent lines for the same values of x.*

Chapter 4

Applications of Derivatives

In this section we introduce higher order derivatives and study two applications of derivatives. The first, curve sketching, permits us to draw a picture that helps us understand the shape and behavior of a curve. When this curve is a mathematical model of a real-word situation, this can be quite useful.

Our second application is *optimization*, the discipline of finding the largest and smallest value a formula takes on in a given interval. This will require us to look a bit at what types of intervals there are. This, in turn raises questions like the following:

What is the largest number smaller than 4?

The problem with that question is that the answer is "there isn't one". Any number $y < 4$ has another number $(y+4)/2$ between it and 4 and so no largest number is possible.

> *In the fall of 1972 (United States) President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection.*
> – **Hugo Rossi**

> *Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.*
> – **W.S. Anglin**

> *...essentially, all models are wrong, but some are useful.*
> – **George E. P. Box**

4.1 Curve Sketching

Curve sketching is making a drawing of the graph of a formula. This drawing may be quite accurate or it may be intentionally inaccurate in some particulars to emphasize some details of the curve.

Definition 4.1 *Suppose that we have a function $y = f(x)$. The* **domain** *of $f(x)$ is the set of real numbers for which $f(x)$ can be computed. The* **range** *of $f(x)$ is the set of real numbers that can result from computing $f(x)$.*

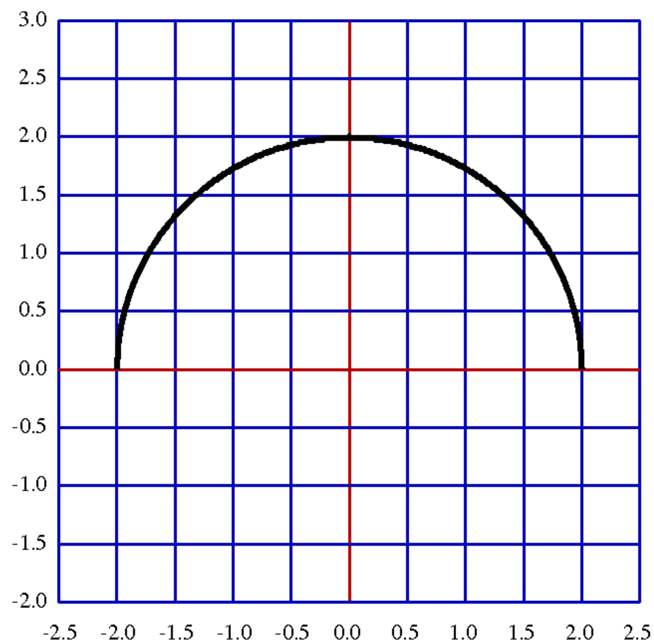
Example 4.1 Domain and range for various functions.

1. *The function* $f(x) = 2x - 1$ *can accept any real value and can produce any real value. Its domain and range are thus "all real numbers".*

2. *The function* $f(x) = x^2$ *can accept any real value, but only positive numbers result from squaring a value. Its domain is "all real numbers" which may be given symbolically as* $-\infty < x < \infty$. *The inequalities are strict because infinity is not a number. The range is all numbers that are at least zero:* $x \geq 0$.

3. *The function* $y = log(x)$ *can only accept positive numbers but can produce any real number (every real number is the log of some other real number). Its domain is thus* $x > 0$ *while its range is* $-\infty < x < \infty$.

4. *The function* $y = \sqrt{4 - x^2}$ *can only accept numbers whose square is 4 or less so the domain is* $-2 \leq x \leq 2$. *This is because negative numbers do not have square roots. If we graph the function:*



it is clear that the range is $0 \leq y \leq 2$.

It can be very helpful to look at a graph when finding the domain and range of a function. The domain is all numbers, less those excluded by impossibilities like square-roots of negatives or division by zero. Finding the range requires a number of trial and error steps but *calculus can be used to reduce the amount of trial and error.*

Students that are familiar with graphing calculators or computer plotting software may wonder why we are working with curve sketching when there are machines that can do curve-completely-accurate-plotting. The main reason is that the sketch can contain, emphasize, or even exaggerate information not available on the raw plot made by a calculator. In addition, learning to sketch curves grants an understanding of what various features in the plot mean.

Example 4.2 Plotting points
The most fundamental tool for graphing a function $f(x)$ *is picking* x *values, letting* $y = f(x)$ *to get the corresponding* y*-values, and then plotting the points on a grid. Start by picking an easy set of points and plugging*

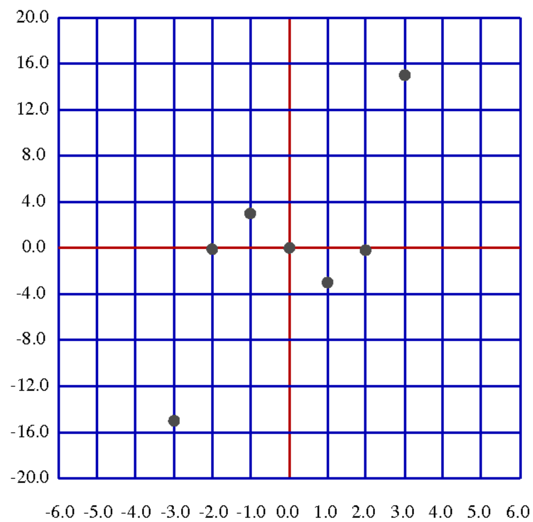
them into the function. If the points you chosen seem to leave the curve ambiguous, then you may want to plot more points.

Problem: *plot points from* $f(x) = x^3 - 4x$.

First set of points:

f(-4)=-48
f(-3)=-15
f(-2)=0
f(-1)=3
f(0)=-0
f(1)=-3
f(2)=0
f(3)=15
f(4)=48

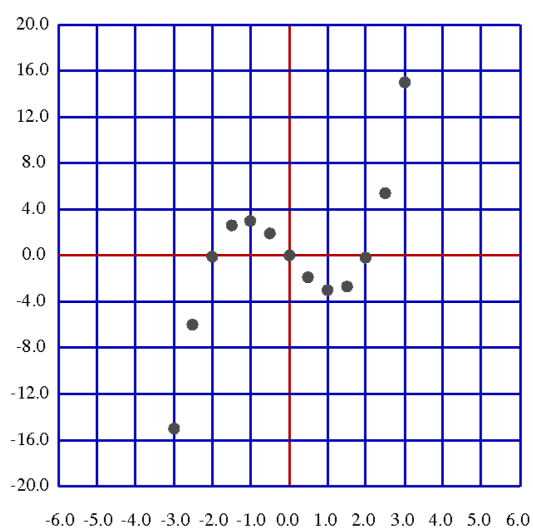
Notice that $f(\pm 4)$ *are too large to fit on the grid: drop them.*



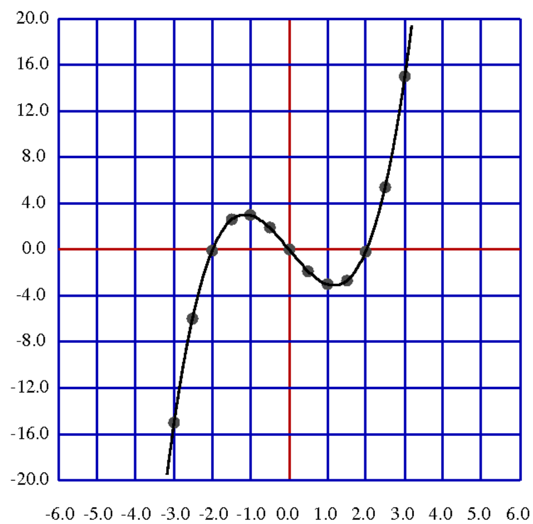
This may not be enough points, and we had to throw out two of them because they we too far off the grid anyway. Let's tighten up the grid a bit.

Next set of points:

f(-3)=-15
f(-2.5)=-5.62
f(-2)=0
f(-1.5)=2.62
f(-1)=3
f(-0.5)=1.88
f(0)=0
f(0.5)=-1.88
f(1)=-3
f(1.5)=-2.62
f(2)=0
f(2.5)=5.62
f(3)=15



These points give us a good sense of the shape of the curve. At this point, we can connect the dots and get a sketch.



We developed a good deal of algebraic machinery to solve for roots - those values of x where a function is zero. For the function in Example 4.2 we can find the roots by factoring:

$$f(x) = x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$$

If we set $x(x-2)(x+2) = 0$ we see that we get roots at $x = 0, \pm 2$ which are among the points. We always plot any roots we know when sketching a curve.

4.1.1 Finding and plotting asymptotes

A function can head off toward infinity in a number of ways. We use asymptotes to characterize these behaviors.

Definition 4.2 *An* **asymptote** *is a line that the graph of a function approaches arbitrarily close to. If such a line is horizontal we call the asymptote a* **horizontal asymptote***; if it is vertical we have a* **vertical asymptote***.*

While there are a few odd exceptions, for our purposes, vertical asymptotes form when the function contains a value of x that leads to division by zero. Horizontal asymptotes are the limit, if one exists, of the function as x approaches $\pm\infty$.

Example 4.3 Finding asymptotes*.*

Problem: *Find the asymptotes of* $f(x) = \frac{2x+3}{4x-2}$*.*

To find vertical asymptote, we simply look for x *where a divide by zero happens. Since* $f(x)$ *is a fraction, we solve the denominator for zero.*

$$\begin{aligned} 4x - 2 &= 0 \\ 4x &= 2 \\ x &= \frac{1}{2} \end{aligned}$$

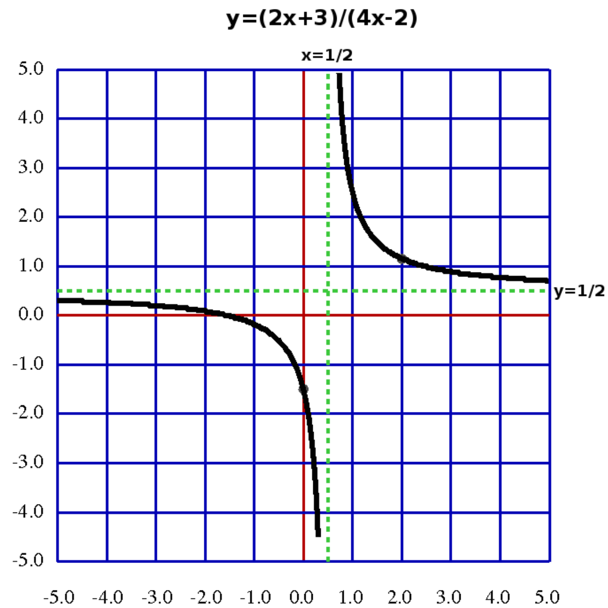
and we have a vertical asymptote at $x = 1/2$*.*

To find horizontal asymptotes we compute the limits of $f(x)$ as x goes to plus or minus infinity. These two limits are often the same, but can be different.

$$\lim_{x \to \infty} \frac{2x+3}{4x-2} = \lim_{x \to \infty} \frac{2+\frac{3}{x}}{4-\frac{2}{x}} = \frac{2+\lim_{x\to\infty}\frac{3}{x}}{4-\lim_{x\to\infty}\frac{2}{x}} = \frac{2+0}{4+0} = \frac{1}{2}$$

and we have a horizontal asymptote at $y = \frac{1}{2}$. Check for yourself that we get $y = 1/2$ if we take the limit as $x \to -\infty$.

Asymptotes are displayed on the graph as dotted lines. Here is the graph for this function:



Look at the function $f(x) = x^3 - 4x$ in Example 4.2. There are no x for which a division by zero happens. The limits as $x \to \pm\infty$ are infinite. That means that $f(x) = x^3 - 4x$ has no asymptote. We now look at a function with no vertical asymptotes but two different horizontal ones.

Example 4.4 Multiple horizontal asymptotes.

Problem: *find the asymptotes of*

$$f(x) = \frac{2e^x - 2}{e^x + 1}$$

Since $e^x > 0$ for all x there is never a divide by zero and so this function has no vertical asymptotes. Lets look for horizontal ones by taking limits.

Start with:

$$\lim_{x \to \infty} \frac{2e^x - 2}{e^x + 1}$$

Divide the top and bottom of the fraction by e^x and we get

$$\lim_{x \to \infty} \frac{2 - \frac{2}{e^x}}{1 + \frac{1}{e^x}}$$

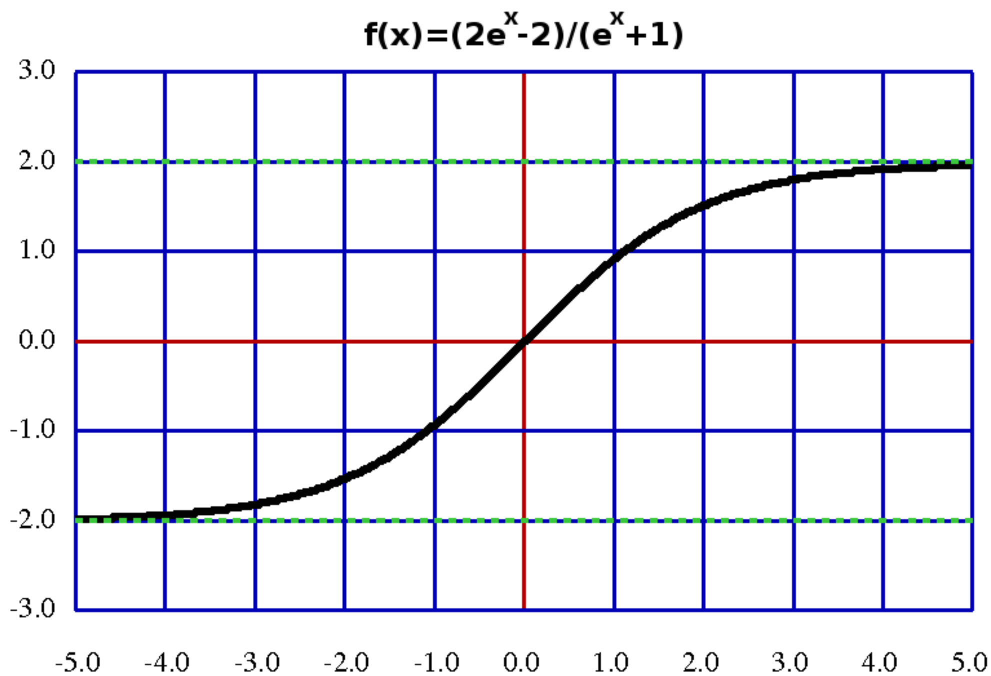
Then, since we know e^x grows without limit we get

$$\frac{2+0}{1+0} = 2$$

and we have the horizontal asymptote for $x \to \infty$. We also know that as $x \to -\infty$ that $e^x \to 0$ so

$$\underset{x \to -\infty}{Lim} \frac{2e^x - 2}{e^x + 1} = \frac{2 \times \underset{x\to-\infty}{Lim} e^x - 2}{\underset{x\to-\infty}{Lim} e^x + 1} = \frac{2 \times 0 - 2}{0+1} = -2$$

and we have the horizontal asymptote for $x \to -\infty$. The horizontal asymptotes are thus $y = \pm 2$. The sketch with these asymptotes looks like this:



The next several examples work on finding asymptotes, providing examples of the various things that can happen. Here are several, potentially useful, facts about asymptotes.

1. There can be zero or more vertical asymptotes.

2. There are always zero, one, or two horizontal asymptotes.

3. A curve must approach a horizontal asymptote *but* it can cross it before it approaches it.

4. A curve *cannot* cross a vertical asymptote.

5. A curve approaches a vertical asymptote in either the positive or negative direction. You can tell which by plugging in points on either side.

6. The approach to a vertical asymptote can be both in the same direction or in opposite directions.

Example 4.5 Multiple vertical asymptotes.

Problem: *find the asymptotes of* $\frac{x}{x^2-4}$

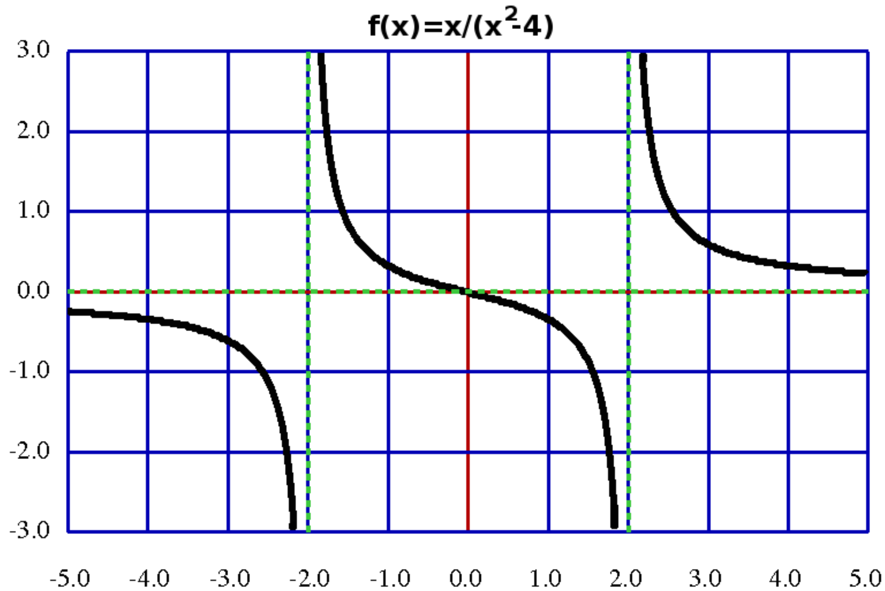
To find vertical asymptotes we solve the denominator for zero:

$$\begin{aligned} x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \\ x &= \pm 2 \end{aligned}$$

giving us two vertical asymptotes. The horizontal asymptote is found by taking the limit

$$\underset{x \to \infty}{Lim} \frac{x}{x^2-4} = 0$$

since the denominator is of higher degree. Here is the graph:



Notice that the curve approaches both its vertical asymptotes in different directions, one approach in the positive direction and one in the negative.

Example 4.6 A curve crossing its horizontal asymptote.

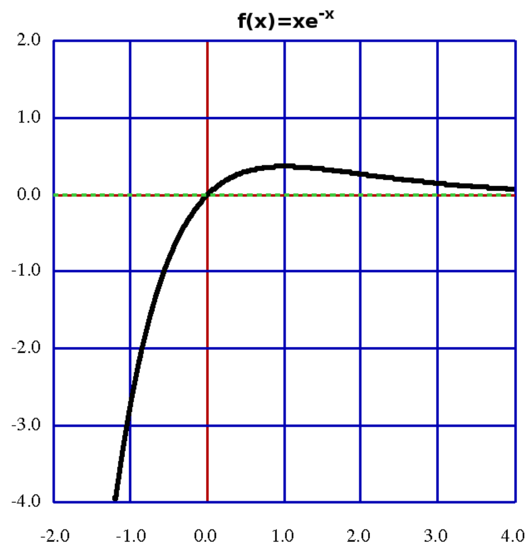
Problem: *find the asymptotes of*

$$xe^{-x}$$

There are no x for which we divide by zero and hence no vertical asymptotes. The horizontal asymptote is found by taking the limit

$$\underset{x \to \infty}{Lim} xe^{-x} = \underset{x \to \infty}{Lim} \frac{x}{e^x} = 0$$

because e^x grows much faster than any power of x. This gives us the horizontal asymptote $y = 0$. Here is the graph:



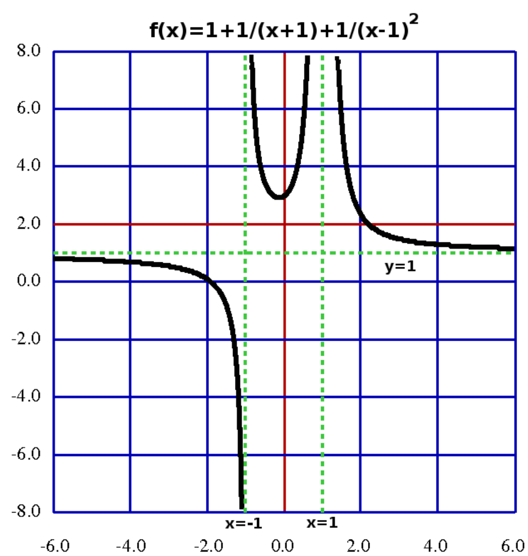
Notice that the curve crosses its horizontal asymptote at (0,0).

Example 4.7 A vertical asymptote with symmetric approaches.

Problem: *find the asymptotes of*

$$1 + \frac{1}{x+1} + \frac{1}{(x-1)^2}$$

We divide by zero at $x = \pm 1$ giving us vertical asymptotes at $x = \pm 1$. Notice that we don't have to combine the expression to spot the vertical asymptotes. Since all the fractions go to zero as x gets larger, in either direction, the horizontal asymptote is clearly $y = 1$. Here is the graph:



Since we are dividing by the first power of $(x + 1)$ the function changes signs after we pass $x = -1$. The other asymptote is associated with dividing by $(x - 1)^2$ which stays positive on both sides of $x - 1$. This is why both asymptotes at $x = 1$ approach in the same direction.

Another Type of Asymptote

Consider the function

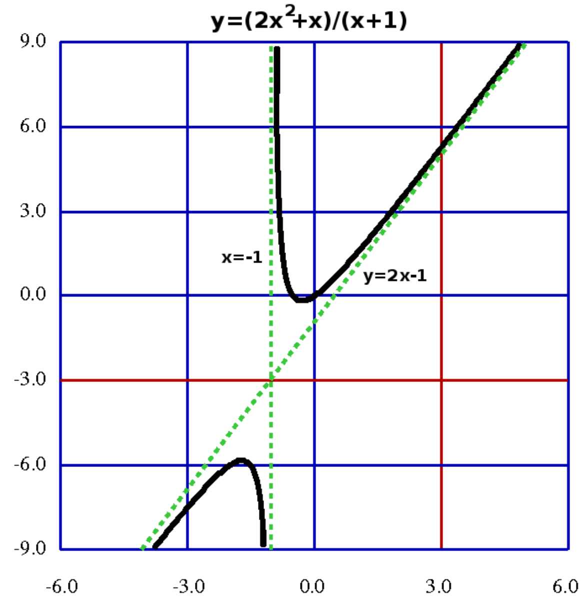
$$f(x) = \frac{2x^2 + x}{x + 1}$$

If we use polynomial long division on the numerator and denominator:

$$\begin{array}{r|l} & \quad\;\; 2x - 1 \\ \hline x+1 & 2x^2 + x \\ & -2x^2 - 2x \\ \hline & \qquad\; -x \\ & \qquad\; x + 1 \\ \hline & \qquad\quad 1 \end{array}$$

We see that $f(x) = 2x - 1 + \frac{1}{x+1}$ From this we see immediately that there is a vertical asymptote at $x = -1$. In addition, since $\frac{1}{x+1}$ will go to zero as x approaches $\pm\infty$ the graph must approach the line $y = 2x - 1$. This is an example of a **diagonal asymptote**, shown in the graph at the right.

Diagonal asymptotes happen in rational functions when the highest power on the top is one higher than the highest power on the bottom.



4.1.2 First derivative information in curve sketching

We already know from Chapter 3 that the derivative $f'(x)$ of a function is the formula for the slope of a tangent line. We can use this slope information to find when the function is increasing and decreasing. Informally a function is increasing when it is sloping uphill, moving left to right. It is decreasing when heading similarly downhill. We want to be precise about which ranges of values a function is increasing or decreasing on, leading us to the following precise definition.

Definition 4.3 *A function is* **increasing** *on a range $a \le x \le b$ if for all $a \le c \le d \le b$ we have that $f(c) < f(d)$. A function is* **decreasing** *on a range $a \le x \le b$ if for all $a \le c \le d \le b$ we have that $f(c) > f(d)$.*

Interval Notation

The notation $a \le x \le b$ is a bit cumbersome. For that reason we now review a more compact notation for intervals of numbers. Suppose that a and b are real numbers with b larger than a. Then we can use either of the notations in the table at the right to describe four different sets of "numbers between a and b".

Longer Notation	Shorter Notation
$a \le x \le b$	$[a, b]$
$a < x \le b$	$(a, b]$
$a \le x < b$	$[a, b)$
$a < x < b$	(a, b)

Definition 4.4 *The numbers* a *and* b *in the intervals above are called the* **endpoints** *of the interval. An interval* $[a, b]$ *that includes its endpoints is called a* **closed interval**. *An interval* (a, b) *that does not include its endpoints is called an* **open interval**.

Example 4.8 Using interval notation.

Longer Notation	*Shorter Notation*	*English Description*
$0 \le x \le 1$	$[0, 1]$	*The set of all numbers between zero and one, inclusive.*
$-2 < x < 2$	$(-2, 2)$	*The set of all numbers between negative two and two, not including* ± 2.
$x \ge 0$	$[0, \infty)$	*The set of all numbers greater than or equal to zero.*
$x > 2$ *or* $x < -2$	$(-\infty, -2) \cup (2, \infty)$	*The set of all numbers greater than two in absolute value. Note that "*$\cup$*" means* union

If you want to glue together multiple intervals, the union symbol $\cup$ *is used.*

Now that we have the shorter interval notation, we can look at how to use the first derivative to accumulate information useful for sketching curves.

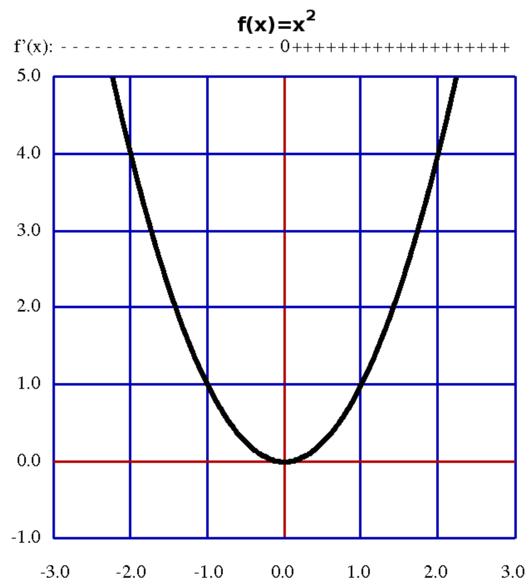
Fact 4.1 *A function* $f(x)$ *is increasing exactly where* $f'(x) > 0$*, it is decreasing exactly where* $f'(x) < 0$ *and when* $f'(x) = 0$ *it is neither increasing nor decreasing.*

Example 4.9 Increasing and decreasing ranges.
Problem: *find the increasing and decreasing ranges for* $f(x) = x^2$.

Strategy: Compute $f'(x)$*, find where it is zero, and test the resulting ranges. This is easy because* $f(x) = x^2$ *is a very simple function. We see that* $f'(x) = 2x$ *which is zero only at* $x = 0$*. Plugging in any negative number to* $f'(x)$ *yields a negative number; plugging in a positive number yields a positive number. We can conclude that* $f(x) = x^2$ *is:*

$$\begin{array}{ll} \textit{Increasing on} & (0, \infty) \\ \textit{Decreasing on} & (-\infty, 0) \end{array}$$

Here is a picture so we can check the results.



Notice that a sign diagram for the first derivative *has been added to the top of the picture.*

The *sign diagram* is a flat summary of the increasing/decreasing behavior of a curve. We now give a couple careful definitions.

Definition 4.5 *A* **critical point** *for a function $f(x)$ consists of any point $(a, f(a))$ or an x value a such that:*

1. $f'(a) = 0$,
2. *$f'(a)$ does not exist, e.g. because of a divide-by-zero.*
3. *The domain of $f(x)$ ends at a, e.g. at zero for $f(x) = \sqrt{x}$.*

Critical points are important because of the following fact.

Fact 4.2 *A function can only change between increasing and decreasing at a critical point. Look at Example 4.9.*

We now have enough bits and pieces to formally define the sign diagram.

Definition 4.6 *A* **sign diagram for the first derivative** *for a function $f(x)$ is a chart that lists the x-coordinate of critical points, with spaces between them, on one line. At each critical point the other line of the chart lists 0 or $*$ as appropriate (* is for critical points other than those where $f'(a) = 0$). Between critical points, the other line of the chart has a chain of $+$ for increasing and $-$ for decreasing. The symbols $\pm\infty$ may optionally be used to border the line with the x values of critical points on it.*

Example 4.10 Sign charts for the first derivative 1) $f(x) = x^2$. *We already know the critical point (0,0) and the increasing and decreasing ranges. This means that the sign chart is:*

$$\begin{array}{ccccc} -\infty & & 0 & & \infty \\ * & - - - - - - - & 0 & + + + + + + & * \end{array}$$

2) $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 2$. *Since this function is polynomial, it will exist for all real numbers and so its critical points are solutions to* $f'(x) = 0$. *Let's find them.* $f'(x) = x^2 - x - 2$ *so:*

$$
\begin{aligned}
x^2 - x - 2 &= 0 \\
(x-2)(x+1) &= 0 \\
&\textit{so} \\
x = 2 \quad &\textit{or} \quad x = -1
\end{aligned}
$$

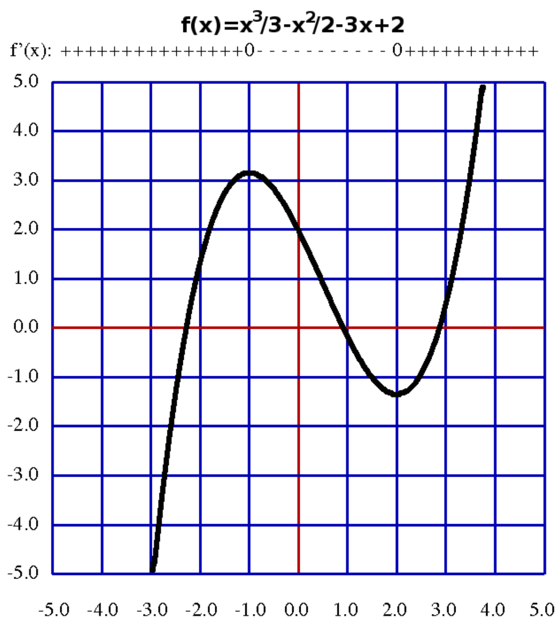
and the critical points are (2,-10/3) and (-1,19/6). The ranges of the sign diagram are thus: $(-\infty, -1)$, $(-1, 2)$, *and* $(2, \infty)$. *Plug a point in each range to* $f'(x)$ *to see if the range is increasing or decreasing.*

$f'(-2) = 4 > 0$ *the function is increasing,*
$f'(0) = -2 < 0$ *the function is decreasing, and*
$f'(3) = 4 > 0$ *the function increasing.*

This makes the sign chart:

$-\infty$		-1		2		∞
*	+ + + + + +	0	− − − − −	0	+ + + + +	*

The sign chart gives a sense of the behavior of the function - increasing since the beginning, decreasing for a short interval, and then back to increasing the rest of the way to infinity. Here is a graph to permit us to check our results:



The increasing and decreasing ranges for this function are:

Increasing	$(-\infty, -1) \cup (2, \infty)$
Decreasing	$(-1, 2)$

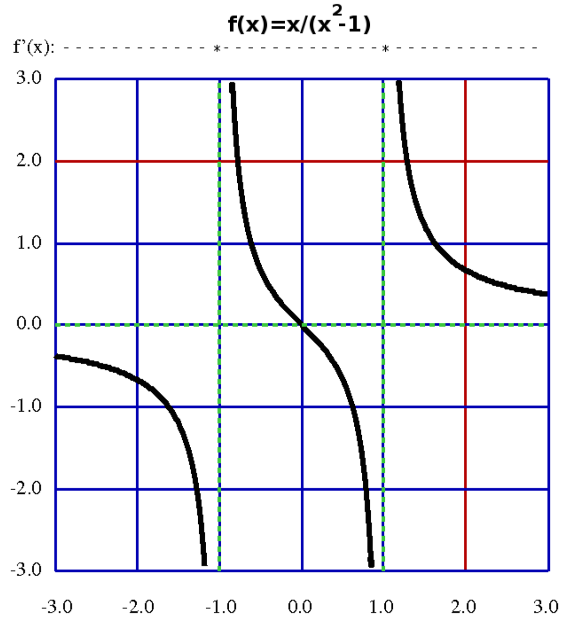
3) $f(x) = \frac{x}{x^2-1}$. *The denominator factors into* $(x-1)(x+1)$ *so there are vertical asymptotes at* $x = \pm 1$*; these are critical points. The first derivative, using the quotient rule, is*

$$\left(\frac{x}{x^2-1}\right)' = \frac{(x^2-1)\cdot 1 - x \cdot 2x}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

Now a fraction is zero only when its numerator is zero at a point where its denominator is not zero: the denominator of $f'(x)$ *is a quadratic with no roots. This means that the vertical asymptotes are the only critical points and the ranges for the sign chart are* $(-\infty, -1)$, $(-1, 1)$, *and* $(1, \infty)$. *For this derivative we do not need to plug in points: the derivative is the negation of two quantities that are always non-negative. This means the derivative is negative wherever it exists and so the function decreases everywhere it exists. This makes the sign chart:*

$-\infty$		-1		1		∞
*	_ _ _ _ _ _	*	_ _ _ _ _	*	_ _ _ _ _	*

*The * in the interior of the sign chart represent vertical asymptotes where the function leaps from* $-\infty$ *to* $+\infty$. *Here is a picture:*



The increasing and decreasing ranges for this function are:

Increasing	*never*
Decreasing	$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Notice that even though the function decreases everywhere it exists, we still need three intervals to deal with the fact the function doesn't exist at ± 1.

4.1.3 Higher order derivatives

In the last section we kept referring to the derivative of a function as the *first derivative*. This was a form of foreshadowing to warn you that there are others: think of it as the mathematical equivalent of ominous music.

Definition 4.7 *The* **second derivative** *of a function is the derivative of the first derivative. It is denoted* $f''(x)$

Example 4.11 Computing second derivatives.

Problem: *find the second derivative of each of the following functions.*

1. $f(x) = x^2$
2. $g(x) = x^3/3 - x^2/2 - 2x + 2$
3. $h(x) = \frac{x}{x^2-1}$

The first derivatives of these were computed in Example 4.10 so we start with the knowledge that:

1. $f'(x) = 2x$
2. $g'(x) = x^2 - x - 2$
3. $h'(x) = -\frac{x^2+1}{(x^2-1)^2}$

We now go on to the second derivatives:

1. $f''(x) = 2,$
2. $g''(x) = 2x - 1,$ *and*
3. $h''(x) = \left(-\frac{x^2+1}{(x^2-1)^2}\right)'$

   Using the quotient rule and simplifying:

$$= -\frac{\left(x^2-1\right)^2 \cdot 2x - \left(x^2+1\right) \cdot 2(x^2-1) \cdot 2x}{\left(x^2-1\right)^4}$$

$$= -\frac{\left(x^2-1\right)^{\cancel{2}1} \cdot 2x - \left(x^2+1\right) \cdot 2\cancel{(x^2-1)} \cdot 2x}{\left(x^2-1\right)^{\cancel{4}3}}$$

$$= -\frac{2x^3 - 2x - 4x^3 - 4x}{\left(x^2-1\right)^3} = \frac{2x^3 + 6x}{\left(x^2-1\right)^3}$$

Notice that polynomial functions get simpler when you take derivatives. Most other functions, excepting $f(x) = e^x$ *which stays the same, tend to get much more complicated when you take their derivatives.*

We formally define the second derivative because we can use it in both curve sketching and in optimization in Section 4.2. There are actually many other derivatives and they are about what you would think. The third derivative is the derivative of the second derivative; the fourth derivative is the derivative of the third; and so on. You can denote higher order derivatives with more prime symbols: $f'''(x)$, $f''''(x)$. There is another notation, especially for when the number of prime symbols gets ridiculous, which is to put the number of derivatives taken as a superscript in parenthesis. Thus $f^{(5)}(x)$ means the fifth derivative of $f(x)$.

At this point we are ready to reveal the role of second derivatives in curve sketching. The second derivative captures the type of curvature a graph has.

Definition 4.8 *A function* $f(x)$ *is said to be* **concave up** *for those values* c *where* $f''(x) > 0$ *and it is said to be* **concave down** *for those values where* $f''(x) < 0$*. See Figure 4.1 for examples of the appearance of these qualities. These qualities are abbreviated* **CCU** *and* **CCD**.

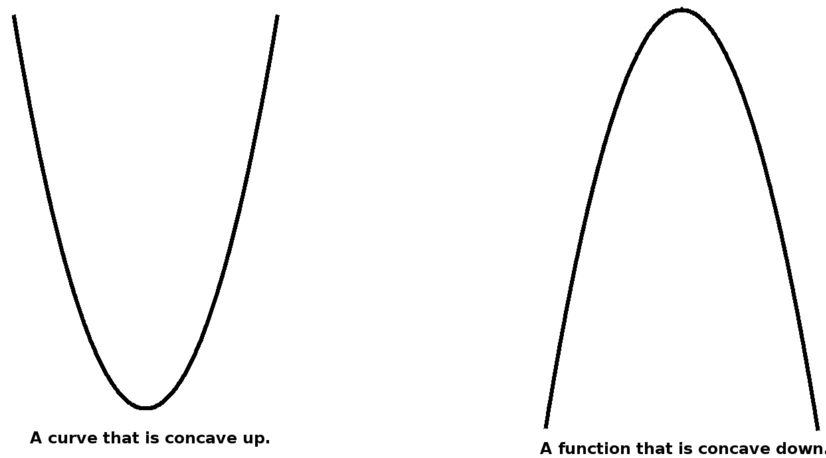


Figure 4.1: Examples of the different types of concavity.

Concavity has similar machinery to increasing and decreasing ranges.

Definition 4.9 *An* **inflection point** *for a function $f(x)$ consists of any point $(a, f(a))$ or any x value a such that:*

1. $f''(a) = 0$,
2. *$f''(a)$ does not exist, e.g. because of a divide-by-zero.*
3. *The domain of $f(x)$ ends at a, e.g. at zero for $f(x) = \sqrt{x}$.*

Just as the function can change from increasing to decreasing at a critical point it can change from concave up to concave down at an inflection point.

Definition 4.10 *A* **sign diagram for the second derivative** *for a function $f(x)$ is a chart that lists the x-coordinate of inflection points, with spaces between them, on one line. At each inflection point the other line of the chart lists 0 or $*$ as appropriate (* is for inflection points not resulting from $f''(a) = 0$). Between inflection points the other line of the chart has a chain of $+$ for concave up and $-$ for concave down.*

Example 4.12 A sign chart for the second derivative.

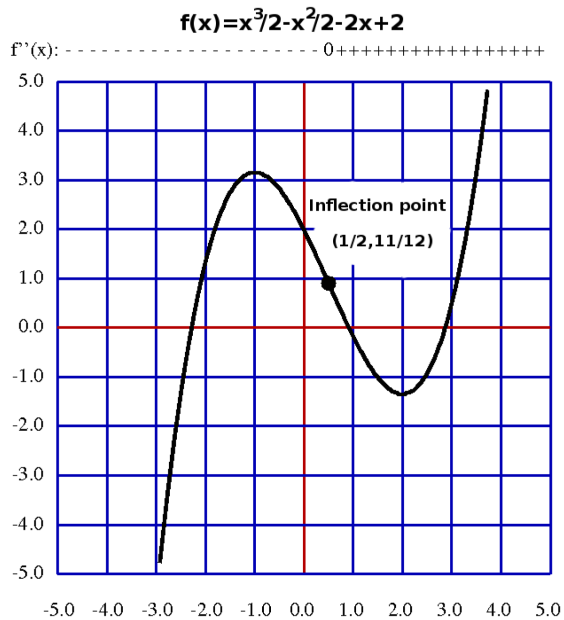
Problem: *make a sign chart for the second derivative for the function*

$$f(x) = x^3/3 - x^2/2 - 2x + 2$$

By now this function should be an old friend from earlier examples in this section. We already know that $f''(x) = 2x - 1$. Solving this equation for zero locates a single inflection point at $x = 1/2$. This makes the sign chart:

$-\infty$		1/2		∞
*	$- - - - -$	0	$+ + + + +$	*

Compare this with the graph:



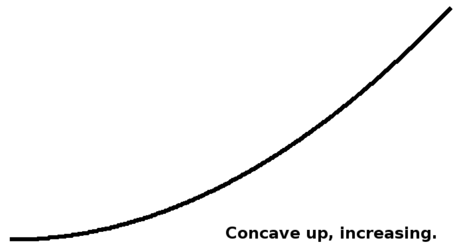
It is easy to see the concave up and concave down regions. The concave up and concave down regions for this function are:

$$\begin{array}{ll} CCU & (1/2, \infty) \\ CCD & (-\infty, 1/2) \end{array}$$

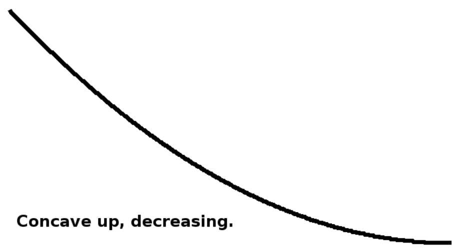
Economic interpretation of derivative information.

Suppose that each of the curves below is the graph of total revenue as a function of the number of items sold. The four curves represent the possible combinations of concavity and increasing/decreasing status.

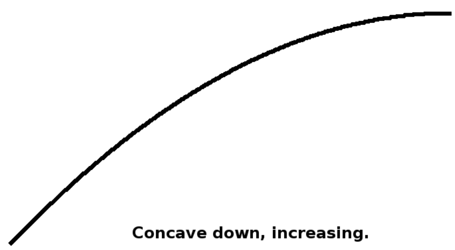
Concave up, increasing.
In this case the revenue is increasing with the number of units made and it is increasing *more* the more units are made. Since this is a money tree, it cannot go on, but it indicates a situation in which expanding production may be a good idea.



Concave up, decreasing.
In this case the revenue is decreasing with the number of units made but is decreasing *less* the more units are made. If this continues, it is hopeless, but it may be this is the initial leg of the curve and it will improve later.



Concave down, increasing.
In this case the revenue is increasing with the number of units made and it is increasing *less* the more units are made. This is typical of a good product where price drops as supply increases. Decisions about production will probably require examining profit, not just revenue.



Concave down, decreasing.
In this case the revenue is decreasing with the number of units made and it is decreasing *more* the more units are made. This situation is typically described with warm, fuzzy adjective like "death spiral".

It is important to embed curves like those above into a larger context before trying to make a decision. Remembering the difference between profit and revenue is also important.

4.1.4 Fully annotated sketches

A fully annotated sketch of a curve is a complete report on the curves behavior. The following are a list of what you should have in an annotated curve sketch, in addition to the sketch itself. It consists of a sketch and a table that reports other facts. The following appear in a fully annotated sketch of a curve:

1. The formula of the function.
2. Any roots, if you can compute or approximate them.
3. Any vertical and horizontal asymptotes.
4. Critical points.
5. The ranges in which the function is increasing and decreasing.
6. Inflection points.
7. The ranges in which the function is concave up and down.
8. Sign charts for the first and second derivatives.
9. A careful sketch.

Example 4.13 Fully annotating a sketch.

Problem: *fully annotate a sketch of*

$$f(x) = x^3 - 4x$$

Since $x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2)$ *the roots of* $f(x)$ *are* $x = 0, \pm 2$. *Since the curve is a non-constant polynomial it has no vertical or horizontal asymptotes. Taking the first derivative and solving it equal to zero we get that*

$$\begin{aligned} 3x^2 - 4 &= 0 \\ x^2 &= \frac{4}{3} \\ x &= \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3} \end{aligned}$$

And we have two critical points at $x = \pm\frac{2\sqrt{3}}{3} \cong \pm 1.15$. *Plugging in* $x = -2, 0,$ *and* 2 *to check if the first derivative is positive or negative we get that* $f(x)$ *increases on* $(-\infty, -2\sqrt{3}/2) \cup (2\sqrt{3}/3, \infty)$ *and decreases on* $(-2\sqrt{3}/2, 2\sqrt{3}/3)$.

Turning to the second derivative we get that $f''(x) = 6x$ *which is zero at* $x = 0$, *yielding a single inflection point at* $x = 0$. *Plugging* $x = \pm 1$ *into the second derivative to see if the curve is concave up or down we get CCU* $(0, \infty)$, *CCD* $(-\infty, 0)$. *We now summarize the results in tabular form and provide a sketch.*

Summary table

Function : $f(x) = x^3 - 4x$

Roots : $x = 0, \pm 2$

Vertical asymptotes : *none.*

Horizontal asymptotes : *none.*

Critical points : $x = \pm\frac{2\sqrt{3}}{3}$

Increasing on : $(-\infty, -2\sqrt{3}/2) \cup (2\sqrt{3}/3,$

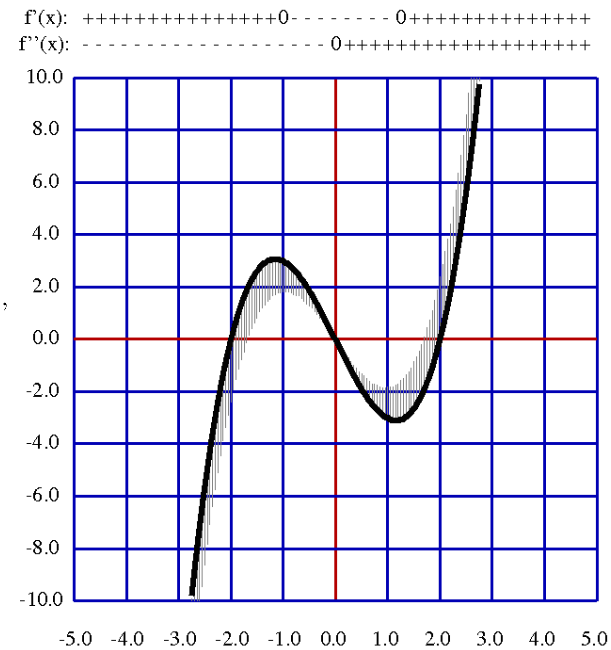
Decreasing on : $(-2\sqrt{3}/2, 2\sqrt{3}/3)$

Inflection points : $x = 0$

CCU on : $(0, \infty)$

CCD on : $(-\infty, 0)$

Notice that the graph has also been shaded with vertical has marks to show where it is CCU and CCD. The sign diagrams for $f'(x)$ and $f''(x)$ appear above the sketch.



Example 4.14 Fully annotating a sketch II.

Problem: *fully annotate a sketch of*

$$f(x) = \frac{1}{x^2 + 1}$$

The numerator of the function is 1 and so is never zero; the function has no roots. There are no x for which we divide by zero and so there are no vertical asymptotes. Computing the limit as $x \to \infty$ we see that

$$\underset{x \to \infty}{Lim} \frac{1}{1 + x^2} = \underset{x \to \infty}{Lim} \frac{\frac{1}{x^2}}{\frac{1}{x^2} + 1} = \frac{0}{0 + 1} = 0$$

yielding a horizontal asymptote at $y = 0$.

Taking the first derivative with the reciprocal rule and solving it equal to zero we get that:

$$\begin{aligned} \frac{-2x}{(x^2+1)^2} &= 0 \\ -2x &= 0 \\ x &= 0 \end{aligned}$$

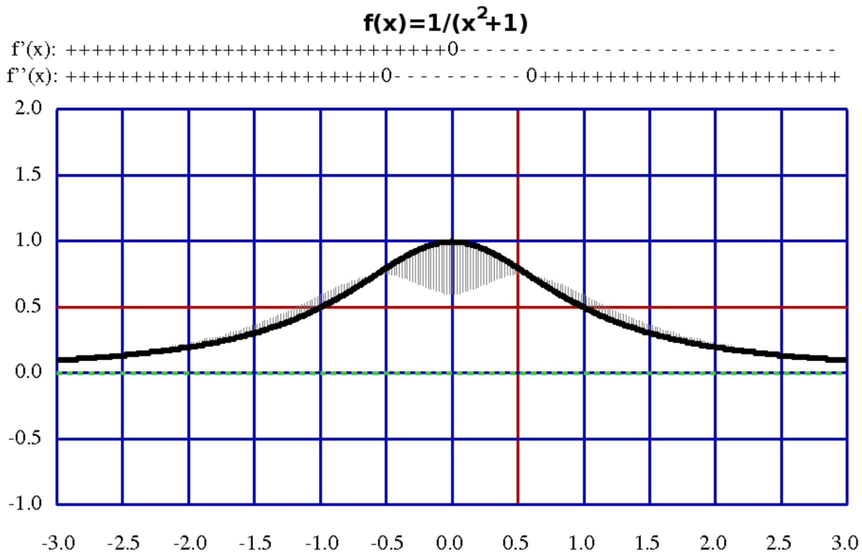
And we have a critical point at $x = 0$. Plugging in $x = \pm 1$ to $f'(x)$ we see the curve increases on $(-\infty, 0)$ and decreases on $(0, \infty)$. Turning to the second derivative we use the quotient rule, simplify, and get that

$$f''(x) = \frac{6x^2 - 2}{(x^2 + 1)^3}$$

Solving the numerator for zero we get:

$$
\begin{aligned}
6x^2 - 2 &= 0 \\
x^2 &= \frac{1}{3} \\
x &= \pm\frac{\sqrt{3}}{3} \cong \pm 0.577
\end{aligned}
$$

demonstrating there are two inflection points at $x = \pm\frac{\sqrt{3}}{3}$. *Plugging in* $x = 0, \pm 1$ *to* $f''(x)$ *we find that the function is CCU on* $\left(-\infty, -\frac{\sqrt{3}}{3}\right) \cup \left(\frac{\sqrt{3}}{3}, \infty\right)$ *and CCD on* $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$. *We now summarize the results in tabular form and provide a sketch.*



Summary table

Function :	$f(x) = \frac{1}{x^2+1}$
Roots :	*none.*
Vertical asymptotes :	*none.*
Horizontal asymptotes :	$y = 0$
Critical points :	$x = 0$
Increasing on :	$(-\infty, 0)$
Decreasing on :	$(0, \infty)$
Inflection points :	$x = \pm\frac{\sqrt{3}}{3}$
CCU on :	$\left(-\infty, -\frac{\sqrt{3}}{3}\right) \cup \left(\frac{\sqrt{3}}{3}, \infty\right)$
CCD on :	$\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

Notice that the graph above has also been shaded with vertical hash marks to show where it is CCU and CCD. The sign diagrams for $f'(x)$ *and* $f''(x)$ *appear above the sketch.*

Example 4.15 Fully annotating a sketch III.

Problem: *fully annotate a sketch of*

$$f(x) = \frac{x}{x^2 - 1}$$

The denominator of the function is x and so is zero at $x = 0$, the sole root. This function has appeared in previous examples that tell us there are vertical asymptotes at $x = \pm 1$ and a horizontal asymptote at $y = 0$.

Borrowing the first derivative from Example 4.10:

$$-\frac{x^2+1}{\left(x^2-1\right)^2}$$

Since the numerator is a quadratic with no roots, we see there are no critical points due to $f'(x) = 0$. There are, however, two critical points at $x = \pm 1$ caused by the division by zero at the vertical asymptotes. Plugging in $x = 0, \pm 2$ to $f'(x)$ we see the curve increases never and decreases on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

The second derivative for $f(x)$ was computed in Example 4.11 and is

$$\frac{2x^3+6x}{\left(x^2-1\right)^3}$$

Solving the numerator for zero we get:

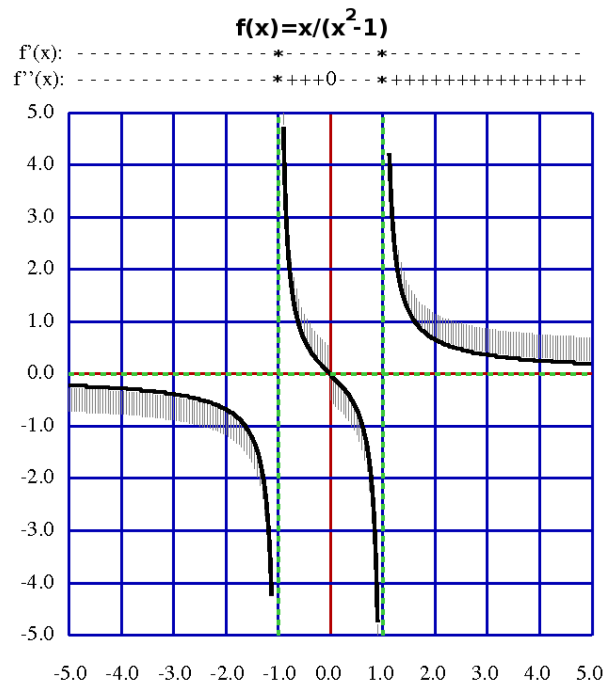
$$\begin{aligned} 2x^3 + 6 &= 0 \\ x(2x^2 + 6) &= 0 \\ x &= 0 \end{aligned}$$

Notice that $2x^2 + 6$ is never zero. We thus have one inflection point due to $f''(x) = 0$ at $x = 0$ and two more at $x = \pm 1$ caused by non-existence of the function. Plugging in $x = 0, \pm 1/2$ to $f''(x)$ we find that the function is CCD on $(-\infty, -1) \cup (0, 1)$ and CCU on $(-1, 0) \cup (1, \infty)$. We now summarize the results in tabular form and provide a sketch.

Summary table

Function :	$f(x) = \frac{x}{x^2-1}$
Roots :	$x = 0$
Vertical asymptotes :	$x = \pm 1$
Horizontal asymptotes :	$y = 0$
Critical points :	$x = \pm 1$
Increasing on :	*never*
Decreasing on :	$(-\infty, -1) \cup (-1, 1)$
:	$\cup (1, \infty)$
Inflection points :	$x = 0, \pm 1$
CCU on :	$(-1, 0) \cup (1, \infty)$
CCD on :	$(-\infty, -1) \cup (0, 1)$

Notice that the graph has also been shaded with vertical has marks to show where it is CCU and CCD. The sign diagrams for $f'(x)$ and $f''(x)$ appear above the sketch.



Exercises

Exercise 4.1 *Find the roots of each of the following functions.*

a) $f(x) = x^2 - 2x - 3$. **b)** $f(x) = x^3 - 7x + 6$. **c)** $f(x) = x^3 - 5x^2 + 7x - 2$. **d)** $f(x) = x^3 + 4x^2 + 6x + 4$.

e) $f(x) = \frac{x^2+x-2}{x^2+1}$. **f)** $f(x) = \frac{x^2+2x+3}{x^2-9}$. **g)** $f(x) = \sqrt{x^3 - 2x^2 - 5x + 6}$.

h) $f(x) = x^2e^{-x} - 16e^{-x}$. **i)** $f(x) = e^{x^2-3x+2}$. . **j)** $f(x) = ln(x^2 - x - 4) - ln(8)$.

Exercise 4.2 *Find all vertical or horizontal asymptotes of the following functions.*

a) $f(x) = \frac{x^2}{x^2-3}$. **b)** $f(x) = \frac{3x^3-2}{x^3-2x^2-x+2}$. **c)** $f(x) = \frac{x^2-4}{x^2-4x+3}$.

d) $f(x) = \frac{e^x+1}{e^x+2}$. **e)** $f(x) = \frac{1}{xe^x-e^x}$. **f)** $f(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$.

Exercise 4.3 *For the intervals or collections of intervals below, make a table like that in Example 4.8 giving the two other methods of describing the numbers.*

a) $-1 \le x \le 5$, **b)** $4 < x < 40$, **c)** $[2, 7)$, **d)** $(-2, -1) \cup (1, 2)$,

e) The set of all real numbers, and **f)** The set of numbers that are at most 5 but strictly more than 2.

Exercise 4.4 *For each of the following functions, give the sign chart for the first derivative.*

a) $f(x) = x^2 + 3x + 2$. **b)** $f(x) = 4 - x - x^2$. **c)** $f(x) = x^3 - 12x + 5$. **d)** $f(x) = x^3 - \frac{3}{2}x^2 - 18x + 7$. **e)** $f(x) = \frac{2x}{2-x}$. **f)** $f(x) = \frac{x^2}{x^2-4}$.

Exercise 4.5 *Using the compact interval notation, give the increasing and decreasing ranges for the function in problem 4.4.*

Exercise 4.6 *Make first derivative sign chart for the following functions, compare them, and explain what's going on in a sentence or two.*

a) $f(x) = x^3$. **b)** $f(x) = x^5$. **c)** $f(x) = x^4$. **d)** $f(x) = x^6$.

Exercise 4.7 *For each of the following first derivative sign charts, sketch a function that could have generated that sign chart. You need not find a formula for the function.*

a)

$-\infty$		2		∞
*	+ + + + +	0	− − − − −	*

b)

$-\infty$		2		∞
*	− − − − −	0	+ + + + +	*

c)

$-\infty$		-3		3		∞
*	− − − − −	0	+ + + + +	0	− − − − −	*

d)

$-\infty$		2		4		∞
*	+ + + + +	0	− − − − −	0	+ + + + +	*

e)

$-\infty$		-1		1		∞
*	+ + + + +	*	− − − − −	*	− − − − −	*

f)

$-\infty$		-3		3		∞
*	+ + + + +	*	+ + + + +	2	− − − − −	*

Exercise 4.8 *Find the second derivative of each of the following function.*

a) $f(x) = x^3 - 3x + 2$. **b)** $f(x) = x^3 - 9x$. **c)** $f(x) = \frac{x}{(x-1)^2}$.

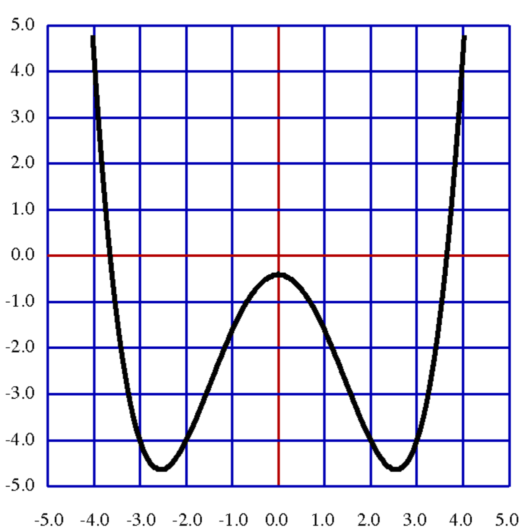
d) $f(x) = \frac{1}{x^2+4}$ **e)** $f(x) = x^2 e^{-x}$. **f)** $\frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Exercise 4.9 *For each of the following functions, do a fully annotated sketch.*

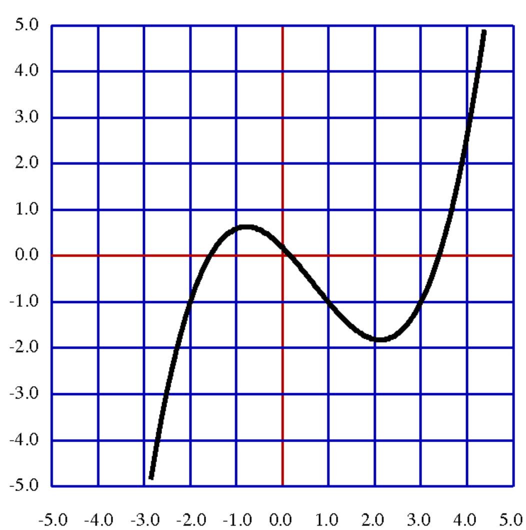
a) $f(x) = x^2 - 2x + 2$. **b)** $f(x) = x^3 - 9x$. **c)** $f(x) = \frac{x^2-4}{x^2-4x+3}$.

d) $f(x) = \frac{1}{x^2-2x+1}$ **e)** $f(x) = x^2 e^{-x} - 16e^{-x}$. **f)** $\frac{e^x - e^{-x}}{e^x + e^{-x}}$.

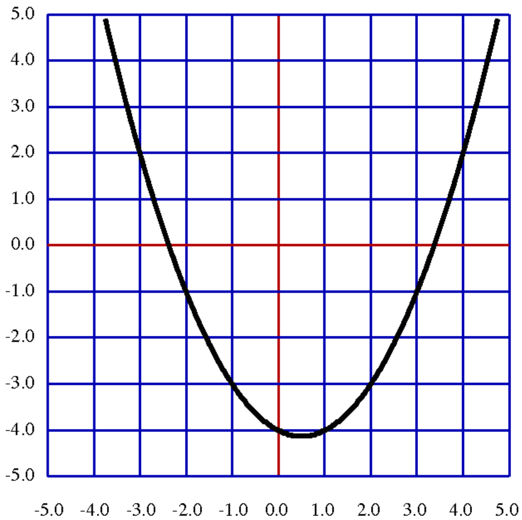
Exercise 4.10 *For each of the following sketches, estimate the critical points, increasing and decreasing ranges, inflection points, and CCU and CCD regions as best you can using the grid supplied.*



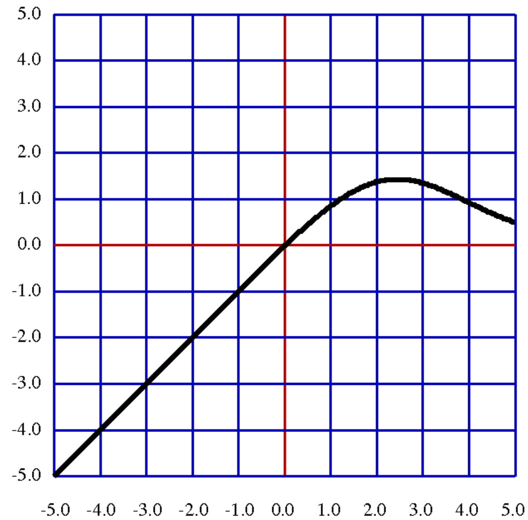
a)



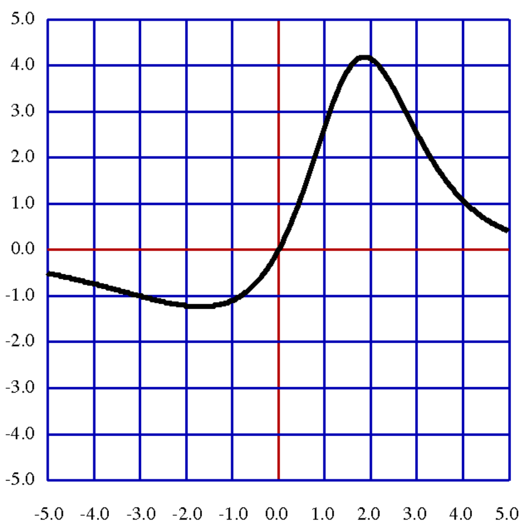
b)



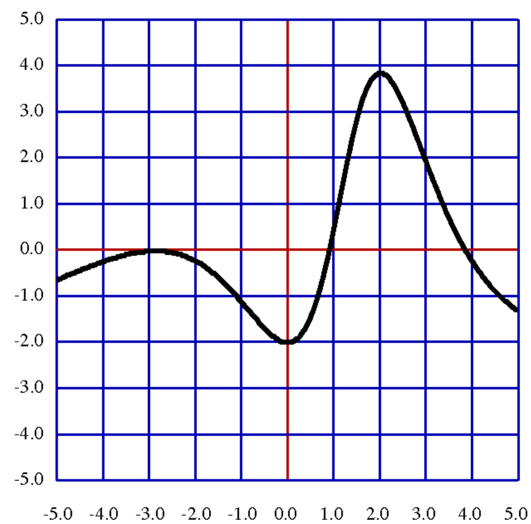
c)



d)



e)



f)

Exercise 4.11 *Do a fully annotated sketch for*

$$\frac{x^2 + x - 8}{x - 2}$$

including the diagonal asymptote.

Exercise 4.12 *Do a fully annotated sketch for*

$$\frac{x^2 - 5x + 3}{1 - x}$$

including the diagonal asymptote.

Exercise 4.13 *Find the third derivative of* $f(x) = xe^x$.

Exercise 4.14 *Find the third derivative of $f(x) = x^2 e^x$.*

Exercise 4.15 *Sketch a function that is increasing on two intervals but never decreasing.*

Exercise 4.16 *Sketch a function with a vertical asymptote at $x = 1$, a horizontal asymptote at $y = 1$ that never gets larger than 2.*

Exercise 4.17 *Sketch a function that has two intervals in which it is increasing and two on which it is decreasing.*

Exercise 4.18 *Sketch a graph that decreases before 2, increases from 2 to 5, and decreases after 5. It should also have a vertical asymptote at 5.*

Exercise 4.19 *Do a fully annotated sketch for*

$$f(x) = \frac{x^3 - 2x + 3}{x^3 - x^2 - x + 1}$$

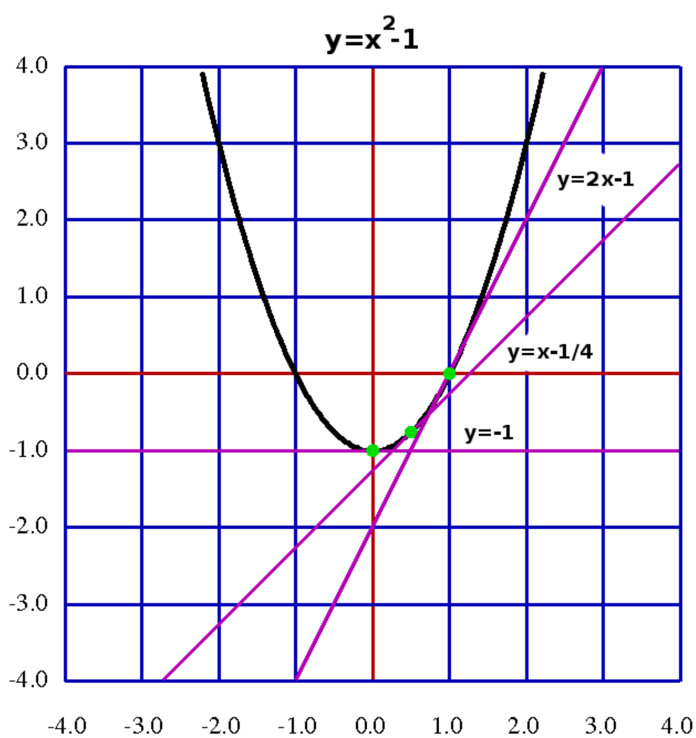
Exercise 4.20 *Do a fully annotate sketch for $f(x) = \sqrt{x^2 + 4}$.*

Exercise 4.21 *Do a fully annotated sketch for $f(x) = x^2 e^{-x}$.*

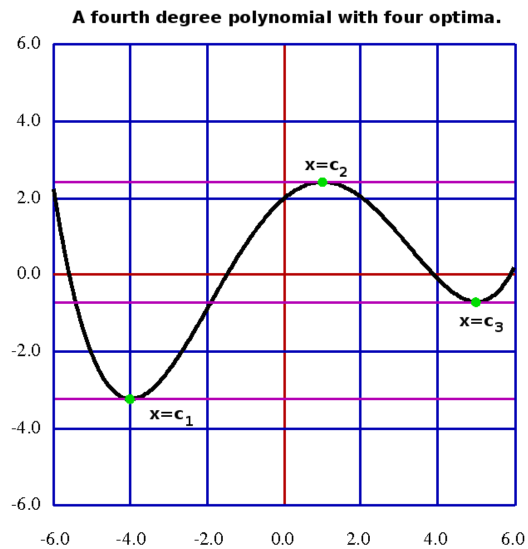
Exercise 4.22 *If $f(x) = p(x)e^x$ where $p(x)$ is a quadratic equation, demonstrate logically that $f'(x) = q(x)e^x$ where $q(x)$ is also a quadratic.*

4.2 Optimization

In Chapter 3 we saw how to find tangent lines to a curve. Examine the three tangent lines to $f(x) = x^2 - 1$ below at $x = 0, \frac{1}{2}, 1$.

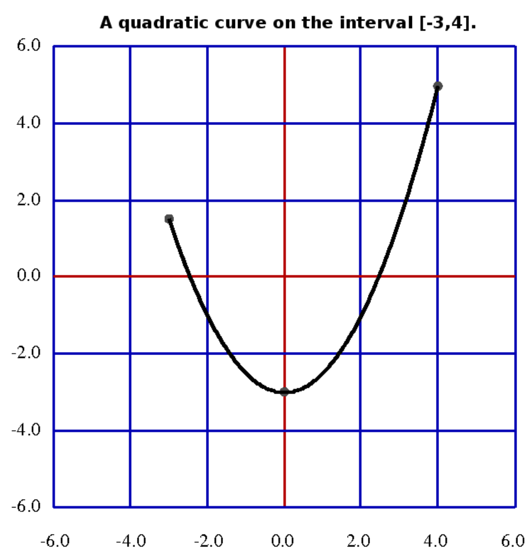


Notice that the tangent line at $x = 0$ is horizontal. It also hits the vertex of the parabola which is its minimum value. This means that the equation $f'(x) = 0$, which locates horizontal tangent lines, can be used to find minimal and maximal values of continuous functions.
Notice that when $f'(x) = 0$ then the curve, whether increasing or decreasing, turns around and goes the other way. The picture below is of a 4th degree polynomial that has three points for which $f'(x) = 0$.



The three horizontal tangent lines are shown in violet.

There is one additional complication to the process of optimization. If the values of x are bounded then the largest or smallest value may *not* happen for some x such that $f'(c) = 0$. It may also happen at a boundary point. Examine the following picture:



The function above is $f(x) = \frac{x^2-6}{2}$ on the compact interval [-3,4]. The lowest point (y-value) is (0,-3) and the highest is (4,5). This means the maximum is 5 and occurs at an endpoint.

Fact 4.3 *Consider the graph of $f(x)$ on the closed interval [a,b]. If $f(x)$ is continuous on the interval then the largest and smallest value of y must occur at a, b, or some value c for which $f'(c) = 0$.*

While we will develop other useful tools that help with calculus based optimization, the preceding fact is the key to the whole affair. When we are solving real world problems, the formula we are optimizing usually do have a closed interval on which they make sense in the real world - a box or can cannot have infinite or negative dimensions, for example, and so restricting the function to an interval is a realistic constraint.

Notice that we can rephrase Fact 4.3 by saying that **optima occur at critical points or endpoints**.

Example 4.16 Verifying a geometry fact by optimization.

Problem: *demonstrate that a rectangle of fixed perimeter* $P = 100cm$ *has its largest area when it is a square.*

A rectangle with height H *and width* W *has a perimeter of* $2H + 2W$ *and an area of* $A = H \times W$. *From this we may deduce that*

$$2H + 2W = 100$$

Solving for W *we see that* $W = 50 - H$ *and so we may substitute into the area formula to remove one of the two variables obtaining* $A = H(50 - H)$. *This means we now have area as a function of the height of the rectangle:*

$$A(H) = 50H - H^2$$

The largest possible height is 50, the smallest 0 so we need to optimize $A(H)$ *on the interval [0,50]. The derivative is*

$$A'(H) = 50 - 2H$$

Solving $A'(H) = 0$ *we obtain* $H = 25cm$. *This means the maximum and minimum values occur at* $H = 0, 25, 50$.

Height	*Width*	*Area*
0cm	*50cm*	$0cm^2$
25cm	*25cm*	$625cm^2$
50cm	*0cm*	$0cm^2$

So the maximum area occurs when $H = W = 25cm$, *and we see the rectangle is a square.*

Now we need some terminology to describe the various types of objects we have learned to locate.

Definition 4.11 *A* **global maximum** *for a function is the largest value it can take on. A* **local maximum** *is a value of* $y = f(c)$ *that is larger that all other nearby values - a* hill top.

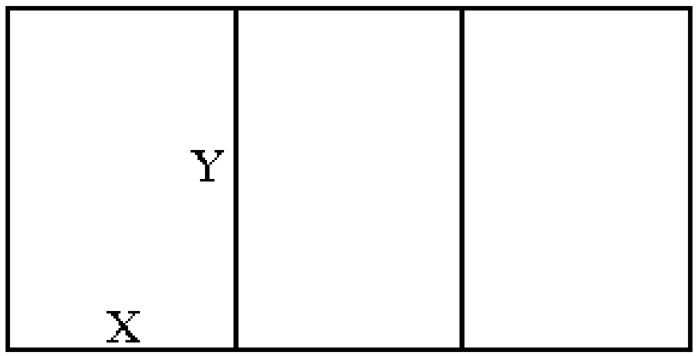
Definition 4.12 *A* **global minimum** *for a function is the smallest value it can take on. A* **local minimum** *is a value of* $y = f(c)$ *that is smaller that all other nearby values - a* valley bottom.

Definition 4.13 *An* **optimum** *(plural* **optima***) is a name used to describe either maxima or minima. Optima can be global or local.*

All the terminology in the definitions above gives us a language to talk about the objects we are working with while performing optimization. Look at the fourth degree polynomial with three horizontal tangents depicted previously. The three horizontal tangents include two local minima, one local maximum, and the global minimum for the function at $x = c_1$. Though it is not obvious from the picture, the function does not have a global maximum because it goes to infinity as x grows.

Application: Optimizing Use of a Resource.

A frequent goal of optimization is to minimize the amount of a resource used to reach a particular goal.



A company is testing three different types of feed to see if they enhance weight gain in chickens. To try and control other variable factors as much as possible, they want the pens for the chickens adjacent, leading to the layout shown above. If each pen must have an area of 600 square meters, what is the minimum length of fence needed? Step by step:

1 The area requirement tells us $XY = 600$.

2 There are 6 pieces of fence of length X and four of length Y. This means that the total amount of fence is $P = 6x + 4Y$. The total amount of fence P is what we want to minimize.

3 Solving the area equation for Y tells us that $Y = \frac{600}{X}$. If we substitute this into the equation for the total amount of fence we get

$$P = 6X + 4\left(\frac{600}{X}\right) = \frac{6X^2 + 2400}{X}$$

4 Next we find the zeros of the first derivative of $P(X) = \frac{6X^2+2400}{X}$:

$$P'(X) = \frac{X \cdot 12X - \left(6X^2 + 2400\right) \cdot 1}{X^2} = 0$$

$$\frac{6X^2 - 2400}{X^2} = 0$$

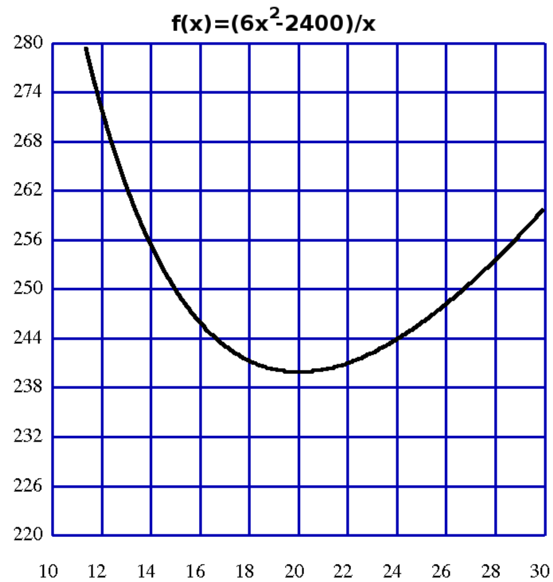
Solving the numerator for zero we get

$$\begin{aligned} 6X^2 &= 2400 \\ X^2 &= 400 \\ X &= \pm 20 \end{aligned}$$

5) Since X is a length, it cannot be negative. We already know, from our geometry example, that long, thin pens would be a waste of fencing so the value $X = 20$ should give us the optimum. If $X = 20m$ then $Y = 30m$ and we see that $P = 6 \times 20 + 4 \times 30 = 240m$ is the least fence we can get away with.

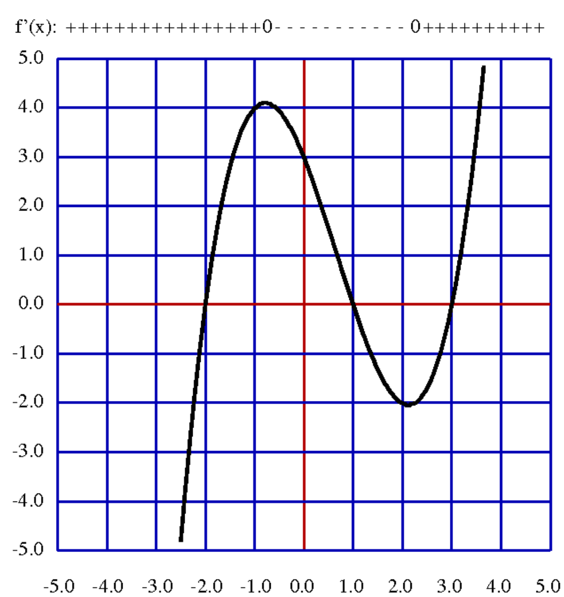
4.2.1 Is it a Maximum or a Minimum?

The boxed application that minimizes the amount of fence to enclose three $600m^2$ pens was not completely rigorous in justifying that $X = 20m$ was a minimum, whether it be local or global. One option is to simply graph the function $P(x) = \frac{6x^2-2400}{x}$:



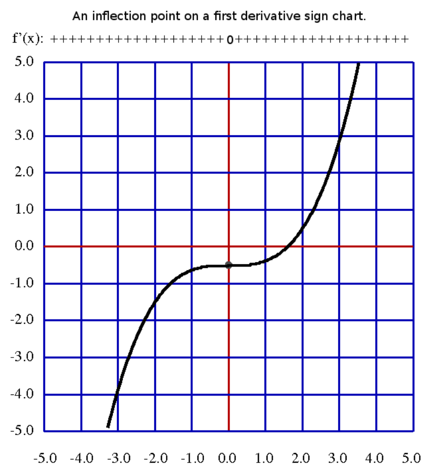
The graph makes it amply clear that setting $X = 20m$ yielded a minimum in the amount of fence used. Graphing a function can be a bit of a production and there are other methods for determining if a given zero of the first derivative is a local maximum or minimum.

Example 4.17 Classifying optima with first derivative a sign chart. *Look at the first derivative sign chart at the top of this graph:*



Notice that $+ + + + + 0 - - - - -$ *tells you a zero of the first derivative is a maximum while* $- - - - - 0 + + + + +$ *tells us it is a minimum. This means we can use sign charts to classify optima as maxima or minima.*

The statement **if A then B** often misleads people into thinking that if **B** is true they somehow know something about **A**. In math this is often not the case. Look at the following graph with a sign chart.

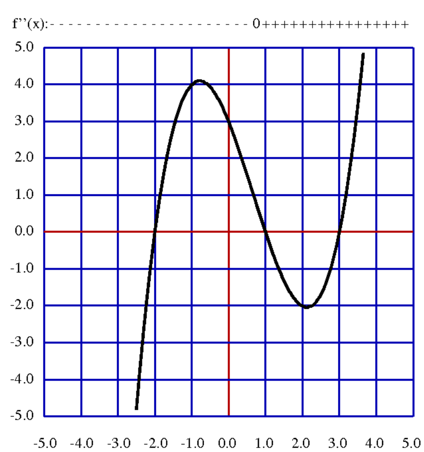


This shows that while optima occur at zeros of the first derivative there is no need for zeros of the first derivative to be optima. The sign chart immediately spots this possibility as sign charts of the form $+ + + + + 0 + + + + +$ or $- - - - - 0 - - - - -$.

Fact 4.4 The first derivative test *Suppose that for a continuous function that* $f'(c) = 0$. *Then:*

1. *If the sign chart near* $x = c$ *is* $+ + + + + 0 - - - - -$ *then* $f(x)$ *has a maximum at* $x = c$.
2. *If the sign chart near* $x = c$ *is* $- - - - - 0 + + + + +$ *then* $f(x)$ *has a minimum at* $x = c$.

Example 4.18 *Look at the second derivative sign chart at the top of this graph:*



Notice that if an optima is a maximum then the curve is CCD and the second derivative is negative. On the other hand, if an optima is a minimum then the curve is CCU and the second derivative is positive. This means that the second derivative can be used to classify optima.

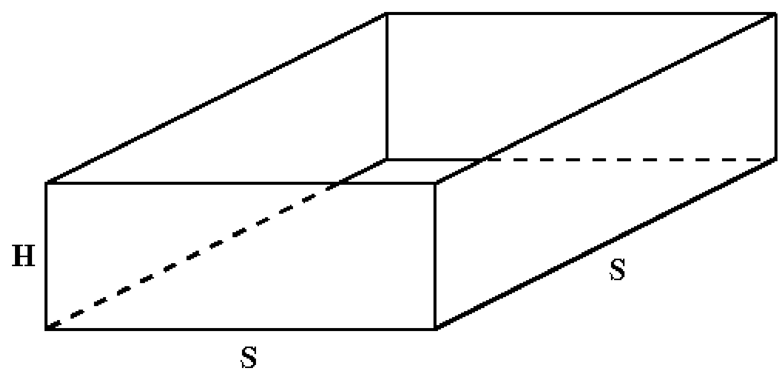
Fact 4.5 The second derivative test. *Suppose that for a continuous function that $f'(c) = 0$. Then*

1. *If $f''(c) < 0$ then $f(x)$ has a maximum at $x = c$.*
2. *If $f''(c) > 0$ then $f(x)$ has a minimum at $x = c$.*
3. *If $f''(c) = 0$ then the test yields no information.*

Now that we have the machinery of optimization developed, we will apply it to a series of material optimization tasks.

Example 4.19 The topless box.

If a square-bottomed cardboard box with no top is to have a volume of one cubic meter then what dimensions minimize the surface area and hence the amount of cardboard required?



The box bottom has an area of S^2 while there are four sides with an area of SH. This means that the surface area, which we are trying to minimize, is

$$A = 4SH + S^2$$

We also have information about the volume of the box:

$$V = HS^2 = 1$$

which tells us that $H = \frac{1}{S^2}$ Substituting in to eliminate one of the variables in the area formula we get that

$$A(S) = 4S\left(\frac{1}{S^2}\right) + S^2 = \frac{4}{S} + S^2$$

We now need to solve the first derivative of the area formula equal to zero.

$$A'(S) = \frac{-4}{S^2} + 2S = \frac{2S^3 - 4}{S^2}$$

which means the critical point(s) we need are solutions to

$$\begin{aligned} 2S^3 - 4 &= 0 \\ S^3 &= 2 \\ S &= \sqrt[3]{2} \cong 1.26 \end{aligned}$$

Checking the sign diagram we see that the sign diagram near $\sqrt[3]{2}$ is $- - - - -0 + + + ++$ and so this is a minimum. Solving for H we get $H = \frac{1}{\sqrt[3]{2}^2} \cong 0.63$ which tells us the dimensions that minimize the use of cardboard are $S = 126cm$ and $H = 63cm$.

After boxes, one of the most common containers are cans, which have a cylindrical shape. A can is a cylinder defined by its height and the radius of its circular top and bottom. We now review the relevant volume and area formulas for cans.

• The area of a circle of radius r is $A = \pi r^2$.

• The surface area of a cylinder of radius r and height h is $A = 2\pi \cdot r^2 + 2\pi \cdot r \cdot h$, because it has a circular top and bottom and sides with can be flattened into a $2\pi \cdot r \times h$ rectangle.

• The volume of a cylinder of radius r and height h is $V = \pi \cdot r^2 \cdot h$.

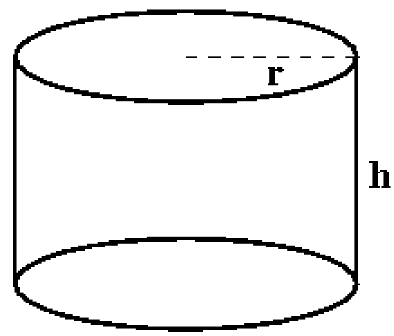
Example 4.20 An optimal can.

Suppose we have a can with a volume of 1000 cubic centimeters. Find the radius and height that minimize the surface area and hence the amount of metal in the can. The relevant equations are:

$$V = \pi \cdot r^2 \cdot h = 1000 \qquad A = 2\pi \cdot r^2 + 2\pi \cdot r \cdot h$$

Solving the first equation for h we get

$$h = \frac{1000}{\pi \cdot r^2}$$



Substituting into the area equation gives us a formula for surface are as a function of radius:

$$A(r) = 2\pi r^2 + (2\pi r) \times \frac{1000}{\pi r^2}$$

$$A(r) = 2\pi r^2 + \frac{2000}{r}$$

$$A'(r) = 4\pi r - \frac{2000}{r^2}$$

Solving this equal to zero we get:

$$\begin{aligned} 4\pi r - \frac{2000}{r^2} &= 0 \\ \frac{4\pi r^3 - 2000}{r^2} &= 0 \\ 4\pi r^3 - 2000 &= 0 \\ r^3 &= \frac{500}{\pi} \\ r &= \sqrt[3]{\frac{500}{\pi}} \cong 5.42cm \end{aligned}$$

Yielding a single critical point. The sign chart near 5.42 is $- - - - - - 0 + + + ++$ *and examining $A(r)$ shows that as the radius or height approach the area grows rapidly. This radius thus yields a minimal area. The corresponding height is $h = \frac{1000}{\pi r^2} \cong 10.84cm$ giving us optimal dimensions for the can.*

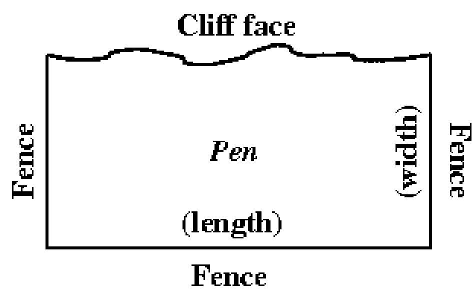
The alert student will have noticed, by this point, that there is a pattern to these optimization problems. We now codify this problem as a set of steps.

Steps for Optimization

1. You start with two equations. One that you want to optimize and another called a **constraint**. Both equations depend on two variables.
2. Solve the constraint for one of the two variables. Typically one of the two choices yields a much easier problem in the subsequent steps: choose wisely.
3. Substitute the solved constraint into the equation you want to optimize to eliminate a variable.
4. Simplify the resulting single-variable equation for the quantity you want to optimize.
5. Use $f'(x) = 0$ techniques to find the optimum value(s) of the remaining variable. This may include using the first and second derivative tests.
6. Plug the optimal value of one variable into the equation you found in Step 2 to get the corresponding value for the other variable. Done.

The next couple examples turns the recourse allocation problem around. They ask "given a fixed amount of a resource, what is the best you can do with it?"

Example 4.21 A pen with a free side.



Suppose that we have 1000m of fence. If we are building a rectangular pen we can enclose a larger area if the fourth side of the pen is a cliff face. Ignoring the roughness of the cliff face, find the largest area we can enclose.

Let L be the length of the pen and W be the width. Then the fact we have 1000m of fence gives us the constraint equation:

$$L + 2W = 1000$$

The area of a rectangle is

$$A = LW$$

. Solving the constraint and substituting in we get

$$L = 1000 - 2W \text{ and thus } A = (1000 - 2W)W$$

So we want to optimize

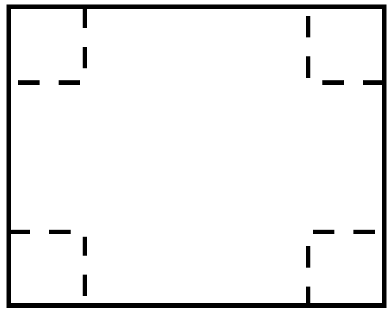
$$A(W) = 1000W - 2W^2$$

$$A'(W) = 1000 - 4W = 0 \text{ so } W = 250m$$

The equation being optimized is a quadratic opening downward so the single critical point is a maximum. The corresponding length is $L = 500$ *which means the largest area we can enclose is*

$$A = LW = 250 \times 500 = 125,000m^2$$

Example 4.22 Making a box by bending up sides.



Starting with a piece of rectangular stock, a flat, open-topped box is made by cutting out four squares of side length s from each corner of the rectangle and then bending up the sides and gluing them. If we are using 40×60 cm stock, what side length for the square maximizes the volume of the box?

The volume of a box is height $\times$ length $\times$ width. This box has height s, length $60 - 2s$, and width $40 - 2s$. This means we can write an equation for the volume without using a constraint:

$$V(s) = s(60 - 2s)(40 - 2s) = 4s^3 - 200s^2 + 2400s$$

The first derivative is $V'(s) = 12s^2 - 400s + 2400$ so we can solve for critical lengths of s with the quadratic formula:

$$s = \frac{400 \pm \sqrt{400^2 - 4 \times 12 \times 2400}}{24} \cong 7.85cm \text{ or } 25.49cm$$

The first derivative is $V''(s) = 24s - 400$ making it easy to use the second derivative test.

$$\begin{aligned} V''(7.85) &= -211.6 < 0 \\ V''(25.49) &= 211.76 > 0 \end{aligned}$$

The second derivative test tells us that $s = 7.85cm$ yields at least a local maximum for box volume. We also know that $0 < s < 20$ (because we must leave some cardboard) but these endpoints yield either a completely flat box with no volume or no remaining cardboard. This means $s = 7.85cm$ is in fact a global maximum with the maximum volume being roughly 8450cc.

Exercises

Exercise 4.23 *Find all horizontal tangents of the following curves.*

a) $f(x) = x^2 + 3x + 5$. **b)** $f(x) = 2x^3 - 3x^2 - 36x + 6$.

c) $f(x) = x^4 - 32x^2 + 5$. **d)** $f(x) = 3x^5 - 65x^3 + 540x - 120$.

e) $f(x) = (2x + 1)e^{-x}$. **f)** $f(x) = x^2e^{-x}$.

g) $f(x) = 4x + \frac{10}{x}$. **h)** $f(x) = 2x + \frac{12}{x}$.

Exercise 4.24 *Find the largest and smallest value of each of the following functions on the given interval. If there is no largest or smallest value, say so.*

a) $f(x) = x^2$ on [-2,1]. **b)** $f(x) = 4x - x^2$ on [-1,6].

c) $f(x) = x^3 - 4x$ on [-2,3]. **d)** $f(x) = 3x^4 - xx^3 - 6x^2 + 24x - 7$ on [-2:2].

e) $f(x) = 2xe^{-x}$ on [0:20]. **f)** $f(x) = 5x^2e^{-x}$ on [0:20].

g) $f(x) = x + \frac{4}{x^2}$ on [0:∞). **h)** $f(x) = 5x + \frac{27}{x}$ on [1:8].

Exercise 4.25 *Apply the first derivative test to each of the functions in Problem 4.23 and use it to classify their optima.*

Exercise 4.26 *Apply the second derivative test to each of the functions in Problem 4.23 and use it to classify their optima.*

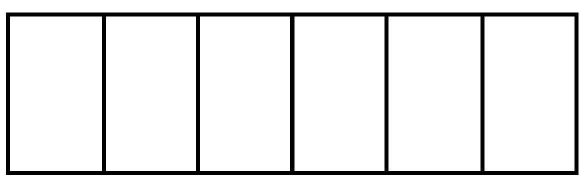
Exercise 4.27 *Each of the following functions models the total profits,in hundreds of dollars, from a manufacturing enterprise as a function of the number n of units manufactured. Find the optimum number of units manufactured. Remember that you can only manufacture positive, whole numbers of units.*

a) $P(n) = \frac{8n}{n^2-80n+1800}$. **b)** $P(n) = \frac{12n}{n^2-120n+2400}$.

c) $P(n) = \frac{n^2}{2n^2-200n+6000}$. **d)** $P(n) = \frac{n^2}{3n^2-160n+4000}$.

e) $P(n) = n^2e^{-n/24}$. **f)** $P(n) = n^2e^{1-n/30}$.

Exercise 4.28 *Suppose that two* $400m^2$ *pens share a common side. Find the minimal length of fence needed to enclose them.*



Exercise 4.29 *Suppose that six* $100m^2$ *pens are arranged in a row, sharing common sides, as shown above. Find the minimal length of fence needed to enclose them.*

Exercise 4.30 *Find the radius and height that minimize the material needed to make an open-topped can with a volume of 400cc.*

Exercise 4.31 *Find the radius and height that minimize the material needed to make an open-topped can with a volume of 1600cc.*

Exercise 4.32 *Suppose that the sides of a swimming pool, that is twice as long as it is wide, are twice as expensive to build as the bottom. If the pool is an open-topped rectangular box that holds* $2000m^3$ *what width, length, and depth minimize the material cost?*

Exercise 4.33 *A rectangular enclosure for Christmas trees by the side of the road has normal chain link fence that costs $40.00 on three sides and, along the side facing the road, taller, stronger security fence that costs $80.00 per foot. If the enclosure is to have* $8000m^2$ *of area what dimensions minimize the cost. Hint: write out the cost as a function of the length and width.*

Exercise 4.34 *A can for frozen juice has a metal top and bottom and cardboard sides. If the metal costs five times as much as the cardboard, find the height and radius that minimize the materials cost of a can that holds 360 cubic centimeters.*

Exercise 4.35 *Some cans have large radius and small height while others have the reverse. Which of these cans correspond to more expensive top-and-bottom materials and which correspond to more expensive side materials? Why?*

Exercise 4.36 *Suppose that we have a rectangle of perimeter 120m. Answer the following questions.*

a) What is the maximum area the rectangle could have?

b) Can it have any positive area smaller than the maximum?

c) What height yields an area of 10 square meters?

d) How many different whole number areas are possible?

Exercise 4.37 *Is there are more efficient way to lay out the pens in Problem 4.29? Explain.*

Exercise 4.38 *If there were seven pens instead of six in Problem 4.37 would that change things? Explain.*

Exercise 4.39 *In this section we saw that, of all the rectangles with the same perimeter, the square is the one has the largest area. If a square and a circle have equal perimeter, which has the larger area?*

Exercise 4.40 *Given the answer to problem 4.39, should you use cylindrical cans or rectangular boxes as containers? Explain your answer.*

Exercise 4.41 *In the introduction to this chapter it says:*

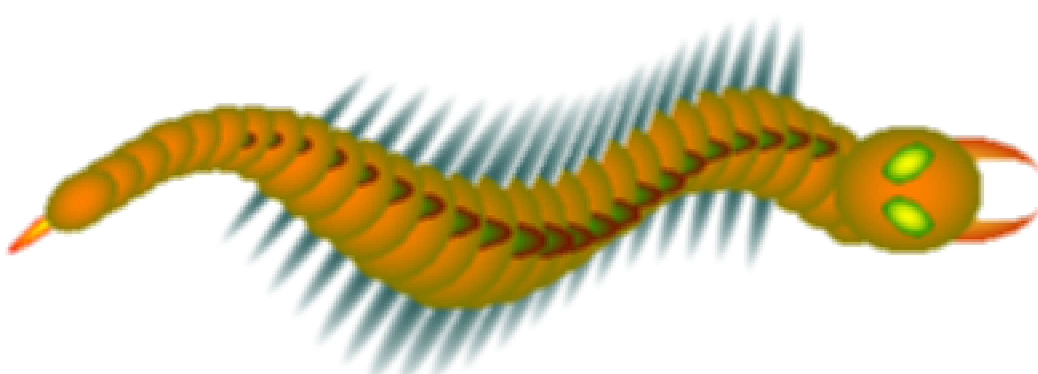
> *What is the largest number smaller than 4?*
>
> *The problem with that question is that the answer is "there isn't one". Any number $y < 4$ has another number $(y + 4)/2$ between it and 4 and so no largest number is possible.*

Demonstrate logically that it is true that

$$y < \frac{y+4}{2} < 4$$

Exercise 4.42 *Show symbolically, not numerically, that the height of the can in Example 4.20 is exactly twice the radius.*

Coming up: a new direction!



Chapter 5

Integrals

God does not care about our mathematical difficulties - he integrates empirically.
–**Albert Einstein**

This is a tricky domain because, unlike simple arithmetic, to solve a calculus problem - and in particular to perform integration - you have to be smart about which integration technique should be used: integration by partial fractions, integration by parts, and so on.
– **Marvin Minsky**

If one looks at the different problems of the integral calculus which arise naturally when one wishes to go deep into the different parts of physics, it is impossible not to be struck by the analogies existing.
– **Henri Poincare**

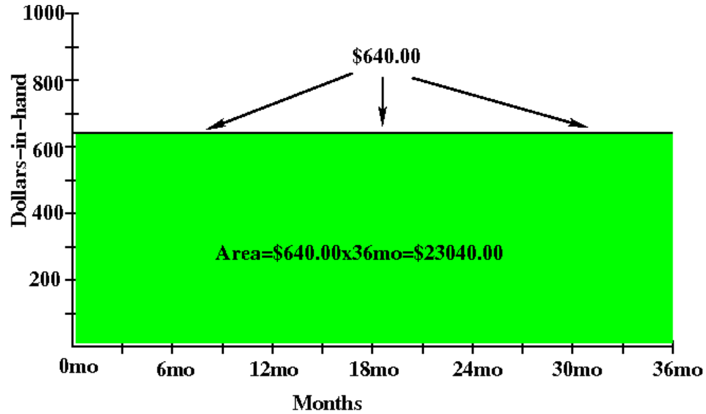
As we've seen in the last two chapters, the derivative of $f(x)$ is the rate at which $f(x)$ is changing. In this chapter we will turn this around and learn to deal with the situation where $f(x)$ *is* the rate at which something is changing. Our goal is to learn to total up that thing and find how much of it there is.

Example 5.1 Constant rates of change

Suppose that you have a rental property that yields $1200.00 per month of income but which costs $560.00 to pay for maintenance, upkeep, insurance, and utilities. Then the rate at which your money-in-hand is changing is $1200.00-$560.00=$640.00/month. What is the total income for the property over 36 months? Since the rate at which money in hand is changing is constant, this is a very easy problem:

$$36 \times \$640.00 = \$23040.00$$

In spite of its simplicity, this problem contains one of the important ideas of the integral calculus. If $f(x) = \$640.00$ *is the rate at which our money-in-hand is changing and we graph* $f(x)$*:*



Then we see that the (green) area under the graph, but above the x-axis is the total money-in-hand after 36 months. Since the area is a rectangle, we don't need calculus to total up the money.

Lets codify this idea in a fact.

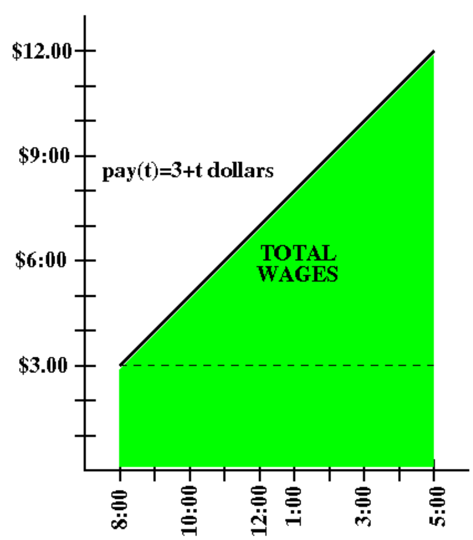
Fact 5.1 *If $f(x)$ is the rate at which a quantity is changing then the area under the graph of $f(x)$ from $x = a$ to $x = b$ is the total amount of the quantity that accumulates between a and b.*

The next example shows how to deal with a rate of change that isn't constant.

Example 5.2 dealing with a non-constant rate of change

Problem: *Suppose that, to keep a group of people working at a task, the organizer says that the pay starts at \$3.00 per hour at 8:00 but goes up continuously at a rate of one dollar per hour until 5:00. That means that the rate of pay at t hours after 8:00 is 3+t dollars/hour. If someone works the whole day, what is their wage?*

Strategy: *Use the fact, start by graphing the rate of pay (which is the rate of change of the wages).*



Notice that the area under the curve can be divided into a rectangle from 8:00 to 5:00 (9 hours wide) and \$3.00 tall and a triangle with a base of 9 hours and a height of \$9.00 (from \$3.00 to \$12.00). Recalling that the area *of a triangle is one-half base times height we get*

$$TOTAL\ \ WAGES = 9 \times 3 + \frac{1}{2} \times 9 \times 9 = 27 + \frac{81}{2} = \$67.50$$

for the full day.

In the first example in this section we see that only the formula for the area of a rectangle is needed. In the second example the area under the curve was a triangle and a rectangle. The more complicated the rate at which the thing we are interested in is changing, the more sophisticated the geometry gets. At this point we want to introduce some calculus that will let us construct the area formula directly.

5.1 Definition of Integrals

Definition 5.1 *Suppose that*

$$F'(x) = f(x)$$

then we say $F(x)$ is an **antiderivative** *of $f(x)$. An older and more traditional name is to say that $F(x)$ is an* **integral** *of $f(x)$.*

The next fact gives us the connection between integrals (or antiderivatives) and the totalling-things-up technique involving finding the area under the curve.

Fact 5.2 Fundamental Theorem of Calculus (Version One)
If $F'(x) = f(x)$ then the area under the graph of $f(x)$ from $x = a$ to $x = b$ is

$$Area = F(b) - F(a)$$

Which means that we can reduce the problem of finding areas under curve (and hence the total change in some quantity) to the problem of finding integrals of functions. There is one odd consequence of this fact: areas that are below the x-axis count as negative "areas". The reason for this is that when a rate of change is below the x axis the rate of change is negative: we are losing rather than gaining ground. Keeping this in mind, lets develop some machinery for finding integrals. Notice that every derivative formula we have now can be turned around to yield an integral formula.

Example 5.3 Finding the integral of x^2

Problem: *Find the integral of x^2.*

We know that

$$\left(x^3\right)' = 3x^2$$

if we divide both sides by three we get that

$$\left(\frac{1}{3}x^3\right)' = x^2$$

so we see that one integral of x^2 is $\frac{1}{3}x^3$. This leads to a big issue with integrals: they are not unique.

Notice

$$\left(\frac{1}{3}x^3 + 2\right)' = x^2 + 0 = x^2$$

Since the derivative of a constant is zero, if $F(x)$ is an integral of $f(x)$ then so is $F(x) + c$ for any possible constant c.

We deal with the issue of that free constant in an integral by using a convention: we write $F(x)+C$ for any integral to remind ourselves that the integral is only known up to a constant. When we are finding an area $F(b)-F(a)$ the constant C appears twice with one positive copy and one negative copy. The copies cancel and the constant disappears. We now give the notation for integrals.

Definition 5.2 Integral notation

When $F'(x)=f(x)$ we say that $F(x)+C$ is the **integral of f(x)** *and we write*

$$F(x)+C=\int f(x)\cdot dx$$

The symbol dx is the **differential of** *x and is used to denote which symbol in the integral is being treated as the variable. This kind of integral, with a "+C" is called an* **indefinite integral**.

If we know the boundaries $x=a$ and $x=b$ we are using to compute an area we write the area as

$$Area=\int_a^b f(x)\cdot dx$$

This sort of integral, with known bounds, is called a **definite integral**.

Example 5.4 Practicing integral notation

We now state the answers to Examples 5.1, 5.2, and 5.3 using integral notation. Example 5.1 is a definite integral:

$$\$23,040=\int_0^{36}\$640.00\cdot d\ months$$

The d months means we are using the integral to total up over a period of months.

Example 5.2 is also a definite integral:

$$\$67.50=\int_{8:00}^{12:00}(3+t)\cdot d\ hours$$

The d hours means the integral is totalling over some number of hours.

Example 5.3 is an indefinite integral:

$$\frac{1}{3}x^3+C=\int x^2\cdot dx$$

We are now ready to start compiling formal rules for performing integrals. Each can be paired with one of our derivative rules, something we will do explicitly at first.

Fact 5.3 Power rule for integrals

Since

$$(x^n)'=nx^{n-1}$$

it is also the case that

$$\frac{1}{n+1}x^{n+1}+C=\int x^n\cdot dx$$

so long as $n\neq-1$. In English this rule is often stated as "add one to the power and divide by the new power". The exception for $n=-1$ is needed to avoid dividing by zero - we fill in this gap in a couple pages with a surprising formula.

There are two mildly special cases. Since $D = Dx^0$ for a constant D we have

$$\int D \cdot dx = Dx + C$$

. Similarly since $x = x^1$ we have

$$\int x \cdot dx = \int x^1 \cdot dx = \frac{1}{2}x^2 + C$$

The linearity rule for derivatives let us extend the power rule for derivatives to all polynomial functions. The same thing happens for integrals.

Fact 5.4 Linearity of Integrals
Suppose that $f(x)$ and $g(x)$ are functions that have integrals and that D is a constant. Then:

1. $\int D \cdot f(x) \cdot dx = D \cdot \int f(x) \cdot dx$

2. $\int (f(x) + g(x)) \cdot dx = \int f(x) \cdot dx + \int g(x) \cdot dx$

3. $\int (f(x) - g(x)) \cdot dx = \int f(x) \cdot dx - \int g(x) \cdot dx$

Example 5.5 Problem: *Find* $\int \left(x^2 - 3x + 5\right) \cdot dx$.

Solution: *Use the linearity rules:*

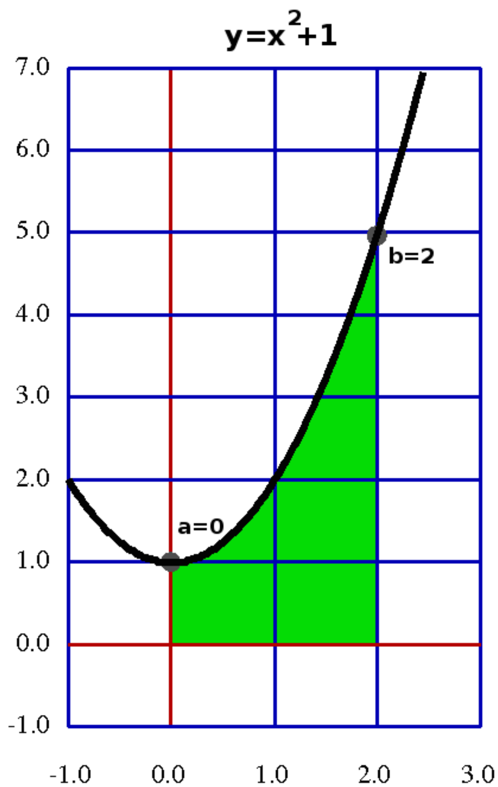
$$\begin{aligned} \int \left(x^2 - 3x + 5\right) \cdot dx &= \int x^2 \cdot dx - 3 \cdot \int x \cdot dx + \int 5 \cdot dx \\ &= \frac{1}{3}x^3 - 3 \cdot \frac{1}{2}x^2 + 5x + C \\ &= \frac{x^3}{3} - \frac{3}{2}x^2 + 5x + C \end{aligned}$$

Notice that, while technically there are three integrals and hence three unknown constants, it is usual to simply lump them all together - any act of indefinite integration requires only one unknown constant at the end.

The linearity rules can also be used with definite integrals.

Example 5.6 Linearity in a definite integral

Find the area under the curve $y = x^2 + 1$ from $a = 0$ to $b = 2$. Note: area "under a curve" means area between the curve and the x-axis



The area we are trying to compute is shown at the left. The corresponding integral is

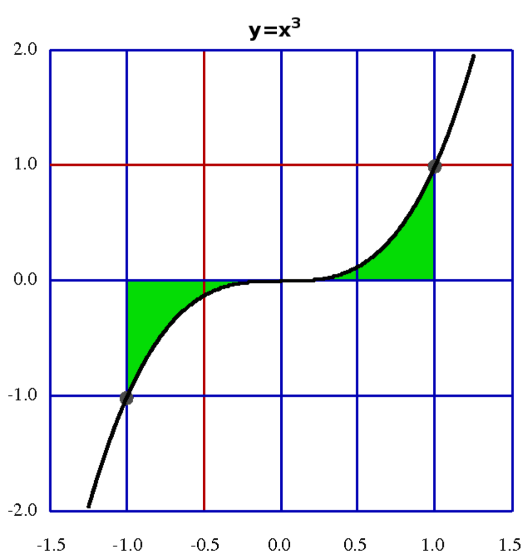
$$\begin{aligned} Area &= \int_0^2 \left(x^2+1\right)\cdot dx \\ &= \int_0^2 x^2\cdot dx + \int_0^2 1\cdot dx \\ &= \frac{1}{3}x^3 + x\Big|_0^2 \\ &= \frac{1}{3}2^3 + 2 - \left(\frac{1}{3}0^3+0\right) \\ &= \frac{8}{3}+2+0 \\ &= \frac{14}{3} \end{aligned}$$

Notice that, once we have performed the integral, the limits of integration ($a = 0$, $b = 2$) are placed on a vertical bar behind the formula before we plug them in to compute $F(b) - F(a)$ as in the fundamental theorem.

The vertical bars, used to hold the limits of integration before we plug them in, are standard notation. Let's do another example.

Example 5.7 Problem: *Find* $\int_{-1}^{1} x^3 \cdot dx$.

Solution: *apply the power rule and the fundamental theorem.*



The area we are trying to compute is shown at the left. The corresponding integral is

$$\begin{aligned} Area &= \int_{-1}^{1} x^3\cdot dx \\ &= \frac{1}{4}x^4\Big|_{-1}^{1} \\ &= \frac{1}{4}1^4 - \frac{1}{4}(-1)^4 \\ &= \frac{1}{4}-\frac{1}{4} \\ &= 0 \end{aligned}$$

Notice that in the diagram the area above and below the x axis are equal. That means the total change, positive and negative, exactly balance out and we get no net change. In fact the integral of an odd power over a plus-minus symmetric area is always zero.

We are now ready to pile up some more integral rules. We know that $(e^x)' = e^x$ and that $(ln(x))' = \frac{1}{x}$. Since integrals are antiderivatives this gives us the following fact.

Fact 5.5 Integrals associated with log and exponential function

1. $\int e^x \cdot dx = e^x + C$
2. $\int \frac{1}{x} \cdot dx = ln(x) + C$

Notice that the second rule above fills in the gap in the power rule for integrals. While the above rules give us integrals associated with the most basic log and exponential functions, they leave us unable to deal with functions like $y = 4^x$ or $y = e^{2x}$. For this we will need a more powerful rule.

Example 5.8 Integration by substitution

Problem: *Find* $\int e^{2x} \cdot dx$.

Solution: *Since we only know how to integrate the exponential of a single variable, let's* make $2x$ *a single variable. Set* $u = 2x$. *Take the derivative of both sides and we get* $u' = 2x'$. *At this point we will use a new fact: the derivative of a variable is a differential; in other words* $x' = dx$ *and* $u' = du$. *This means that*

$$du = 2dx$$

$$\frac{1}{2}du = dx$$

So if we substitute u for x everywhere we can we get:

$$\begin{aligned} \int e^{2x} \cdot dx &= \int e^u \cdot \frac{1}{2}du \\ &= \frac{1}{2}\int e^u \cdot du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{2x} + C \end{aligned}$$

and we have the integral. The key is to make a substitution that turns the problem of interest into one we can do.

The formal name for this technique is *u-substitution*. The reason it is called substitution is obvious, the u comes from the traditional variable name. Let's do another example.

Example 5.9 Integration by substitution, again

Problem: *Find* $\int x(x^2 + 1)^4 \cdot dx$.

Solution: *Since the derivative of* $x^2 + 1$ *is* $2x$ *the quantity we are trying to integrate looks a lot like (a constant multiple of) the derivative of* $(x^2 + 1)^5$. *This suggests that the substitution* $u = x^2 + 1$ *might turn this problem into a simple power-rule integral. If* $u = x^2 + 1$ *we have that* $du = 2x \cdot dx$.

The problem contains x *and* dx *so solve for* $x \cdot dx$ *obtaining* $\frac{1}{2}du = x \cdot dx$. *Now substitute and perform the integral:*

$$\begin{aligned} \int x(x^2+1)^4 \cdot dx &= \int (x^2+1)^4 \cdot x \cdot dx \\ &= \int u^4 \cdot \frac{1}{2} du \\ &= \frac{1}{2} \int u^4 \cdot du \\ &= \frac{1}{2} \left(\frac{1}{5} u^5 \right) + C \\ &= \frac{1}{10} u^5 + C \\ &= \frac{1}{10} \left(x^2+1 \right)^5 + C \end{aligned}$$

and we have the integral. You can verify the integral by computing

$$\left(\frac{1}{10}(x^2+1)^5 \right)'$$

and seeing if it simplifies to $x(x^2+1)^4$.

It is a good idea, for all but the simplest integrals, to check them by taking the derivative of your answer and check to make sure that it simplifies to the expression you originally integrated. Don't freak out if the derivative of your answer is not the thing you started with right away; sometimes a good deal of algebra is needed to connect the dots.

Steps for u-substitution

1. Look at the integrand (thing being integrated), think what sort of thing it might be the derivative of, and choose a tentative expression to be u.
2. Compute $du = u'$.
3. Substitute and see if all the parts of the integrand can be turned into expressions involving u. This may require some algebraic manipulation.
4. If you *cannot* make the algebra work out, go back to step 1.
5. Check and see if you can do the new integral. If you can, do it. If not, go to step 1.

Let's do an integral where algebraic manipulation is needed to make the substitution work.

Example 5.10 More practice with u-substitution

Problem: *Find*

$$\int \frac{x+1}{x-1} \cdot dx$$

Solution: *The annoying part of this is that* $x - 1$ *is not just a single variable. If it were, we could break up the fraction into two integrals, both of which we could do. This means that a logical choice is* $u = x - 1$. *If*

$u = x - 1$ then $du = dx$ and we still need to deal with $x + 1$. But if $u = x - 1$ we can add two to both sides and get $u + 2 = x + 1$. Now we can substitute:

$$\begin{aligned}
\int \frac{x+1}{x-1} \cdot dx &= \int \frac{u+2}{u} \cdot du \\
&= \int \left(\frac{u}{u} + \frac{2}{u}\right) \cdot du \\
&= \int 1 \cdot du + 2 \cdot \int \frac{1}{u} \cdot du \\
&= u + 2ln(u) + C \\
&= x - 1 + 2ln(x-1) + C
\end{aligned}$$

and we have the integral. The difficult step is finding u. Sadly there is no general rule - only lots of examples and skill that increases with practice.

The next example is a little more direct, and variations on it are a classical source of exam problems. Hint! Hint!

Example 5.11 Even more practice with u-substitution

Problem: *Find*

$$\int \frac{x}{x^2+1} \cdot dx$$

Solution: *Since $\left(x^2+1\right) = 2x$ it's fairly obvious that $u = x^2 + 1$ will work. We see $du = 2x \cdot dx$ so that $\frac{1}{2}dx = x \cdot dx$ and we are ready to substitute:*

$$\begin{aligned}
\int \frac{x}{x^2+1} \cdot dx &= \int \frac{1}{x^2+1} \cdot x \cdot dx \\
&= \int \frac{1}{u} \cdot \frac{1}{2}du \\
&= \frac{1}{2} \int \frac{1}{u} \cdot du \\
&= \frac{1}{2}ln(u) + C \\
&= \frac{1}{2}ln(x^2+1) + C
\end{aligned}$$

and we have the integral.

Exercises

Exercise 5.1 *Do the following indefinite integrals. Remember to include everything we should see.*

a) $f(x) = \int x^5 \cdot dx$. **b)** $f(x) = \int x^6 \cdot dx$. **c)** $f(x) = \int \left(x^3 - 2x^2 + 5x - 7\right) \cdot dx$.

d) $f(x) = \int \left(x^4 + x^3 + 2x^2 + x + 1\right) \cdot dx$. **e)** $f(x) = \int ((x-1)(x-2)(x-3)) \cdot dx$.

f) $f(x) = \int \left((x^4 - 5)(x^4 + 5)\right) \cdot dx$.

Exercise 5.2 *Do the following definite integrals.*

a) $f(x) = \int_0^1 x^5 \cdot dx$. **b)** $f(x) = \int_{-1}^1 x^6 \cdot dx$. **c)** $f(x) = \int_0^1 \left(x^3 - 2x^2 + 5x - 7\right) \cdot dx$.

d) $f(x) = \int_0^2 \left(x^4 + x^3 + 2x^2 + x + 1\right) \cdot dx$. **e)** $f(x) = \int_1^2 ((x-1)(x-2)(x-3)) \cdot dx$.

f) $f(x) = \int_{-2}^2 \left((x^4 - 5)(x^4 + 5)\right) \cdot dx$.

Exercise 5.3 *Do the following indefinite integrals, using the technique of substitution.*

a) $f(x) = \int 2x \cdot \left(x^2 - 3\right)^2 \cdot dx$. **b)** $f(x) = \int 3x^2(1 + x^3)^6 \cdot dx$. **c)** $f(x) = \int \frac{x^2}{2x^3 - 4} \cdot dx$.

d) $f(x) = \int \frac{3x^2+2x+1}{x^3+x^2+x+1} \cdot dx$. **e)** $f(x) = \int \frac{dx}{x \cdot ln(x)} \cdot dx$. **f)** $f(x) = \int \frac{e^x}{e^x+2} \cdot dx$.

Exercise 5.4 *Do the following indefinite integrals, using the technique of substitution. In some cases you may want to simplify the expression algebraically before performing the substitution.*

a) $f(x) = \int \frac{x}{x+1} \cdot dx$. **b)** $f(x) = \int \frac{2x+1}{x-1} \cdot dx$. **c)** $f(x) = \int \frac{x^3}{x^2+1} \cdot dx$.

d) $f(x) = \int \frac{x^3+x+1}{x-1} \cdot dx$. **e)** $f(x) = \int \frac{e^x+1}{e^x-1} \cdot e^x \cdot dx$. **f)** $f(x) = \int \frac{e^x+2}{2e^x+3} \cdot e^x \cdot dx$.

Exercise 5.5 *Compute*

$$\int \frac{e^x}{\left(e^x + 4\right)^3}$$

Exercise 5.6 *Compute*

$$\int \frac{e^x}{\left(e^x - 1\right)^5}$$

Exercise 5.7 *Compute*

$$\int_0^{ln(5)} e^{-x} \cdot dx$$

Exercise 5.8 *Compute*

$$\int_0^{ln(3)} xe^{-x^2} \cdot dx$$

Exercise 5.9 *Compute*

$$\int_1^3 \frac{1}{x \cdot ln(x)} \cdot dx$$

Exercise 5.10 *Compute*

$$\int_2^4 \frac{ln(x)}{x \cdot ln(x)^2 + x} \cdot dx$$

Exercise 5.11 *Find the formula, in general, for*

$$\int \frac{ax + b}{cx + d} \cdot dx$$

Exercise 5.12 *Find the formula, in general, for*

$$\int e^{cx} \cdot dx$$

where c is a constant.

Exercise 5.13 *Verify, by taking a derivative, that*

$$\int ln(x) = x \cdot ln(x) - x + C$$

Exercise 5.14 *Verify, by taking a derivative, that*

$$\int x^2 e^x = \left(x^2 - 2x + 2\right) e^x + C$$

Exercise 5.15 *Find the formula, in general, for*

$$\int c^x \cdot dx$$

Hint: use substitution after converting c^x into an expression involving the number e to some power.

Exercise 5.16 *Using the information in Problem 5.13, find a formula for*

$$\int log_b(x) \cdot dx$$

Exercise 5.17 *For $n \geq 0$ find the general formula for*

$$\int_{-a}^{a} x^n \cdot dx$$

Hint: it matters if n is odd or even so you will need a split rule formula.

Exercise 5.18 *First simplify the expression $\frac{1}{x-1} - \frac{1}{x+1}$ and then compute*

$$\int \frac{1}{x^2 - 1} \cdot dx$$

Exercise 5.19 *Show logically that if $p(x)$ is a polynomial of degree n then*

$$\int p(x)e^x \cdot dx$$

is of the form $q(x)e^x + C$ where $q(x)$ is also a polynomial of degree n. Hint: remember that integrals are derivatives done in reverse.

Exercise 5.20 *Since integrals are anti-derivatives, each integration rule is the reverse of some derivative rule. In some cases the derivative rule is so obscure that it doesn't have a name. The integration technique u-substitution, however, is the reverse of a famous derivative rule. Which one? Please explain.*

5.2 Applications of Integrals

An integral is capable of totalling a continuously changing quantity. The next step is to try to find things to total that cannot be done in some simpler way. Example 5.1 showed that when you have a constant rate of change the "integral" to total up the change is just multiplying rate of change by the elapsed time. The earliest example in these notes of a quantity with a non-constant rate of change is the application on page 70: continuously compounded interest.

Example 5.12 Justifying the continuous interest formula

Interest on an account is paid as a fraction of the money present. That means that the rate of change of an interest-bearing account is a fraction c of the current balance. In Chapter 2 we dealt with the case where the interest is paid at specific times and mentioned that continuous interest is exponential. With integration we can demonstrate why this is so.

Suppose that $b(t)$ is the balance in an account at a time t and that the interest rate is a fraction r of that balance. Then the rate at which the account is changing is $r \cdot b(t)$. That means that

$$b'(t) = r \cdot b(t)$$

Let's solve this equation with integration.

$$\begin{aligned} b'(t) &= r \cdot b(t) \\ \frac{b'(t)}{b(t)} &= r \\ \int \frac{b'(t)}{b(t)} \cdot dt &= \int r \cdot dt \end{aligned}$$

At this point set $u = b(t)$ so that $du = b'(t)$. Then:

$$\begin{aligned} \int \frac{1}{u} \cdot du &= \int r \cdot dt \\ ln(u) &= rt + C \\ e^{ln(u)} &= e^{rt+C} \\ u &= e^{rt} \times e^{C} \\ b(t) &= e^{rt} \times e^{C} \end{aligned}$$

Set $A = e^C$ to emphasize that e^C is itself a single constant and we get

$$b(t) = Ae^{rt}$$

It turns out that A is the initial balance of the account ($b(0) = ae^0 = A$)and we see that continuously compounded interest can be demonstrated to be an exponential function.

An integral can be used to total almost anything. In the next example we use it to total losses from a leaking tank. Note that in the middle of the problem, the solution of the integral turns out to be a quadratic equation, at which point we smoothly shift to using the quadratic equation.

Example 5.13 Total losses from a leaking tank

Problem: *Suppose that a storage tank leaks 3.6+2t liters of material per hour where t is the duration since the last time the seals on the tank are tightened. The losses increase as vibration and use loosen the tank's seals. If we want to keep total losses between maintenance cycles down to 150 liters, how often must the seals be tightened?*

Solution: *The integral of losses must be 150L or less, so we solve for the number of hours h until the losses* *reach 150L.*

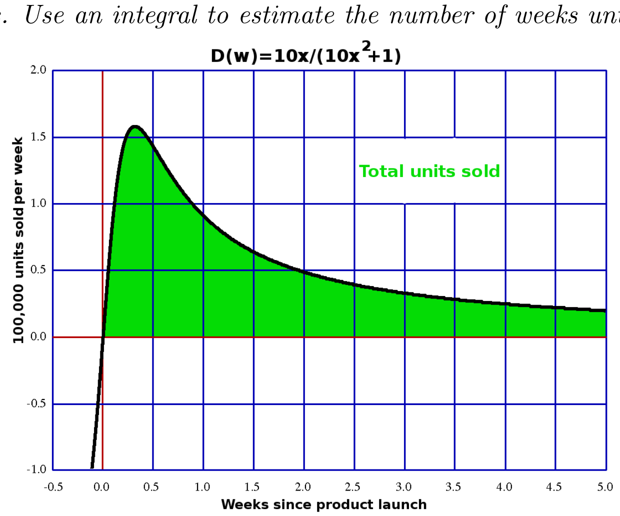
$$\begin{aligned}
150 &= \int_0^h (3.6+2t)\,dt \\
&= 3.6t + 2\times\frac{1}{2}t^2\Big|_0^h \\
&= 3.6t + t^2\Big|_0^h \\
&= 3.6h + h^2 - 0 - 0 \\
150 - 3.6h - h^2 &= 0 \\
\textit{quadratic} &;\quad a=-1,\ b=-3.6,\ c=150 \\
h &= \frac{3.6 \pm \sqrt{3.6^2 - 4(-1)(150)}}{-2} \\
h &= \frac{3.6 \pm \sqrt{612.96}}{-2} \\
h &= \frac{\sqrt{612.96} - 3.6}{2} \\
&\cong 10.58\ \textit{hours}
\end{aligned}$$

So the seals need to be tightened every 10 hours or so. Might be time to invest in a better tank. Notice that we discarded the negative root of the quadratic $150 - 3.6h - h^2 = 0$ *because common sense tells us that negative times are not of interest.*

In the next example we will use a continuously updated estimate of sales per week to compute the number of weeks until 500,000 units are sold.

Example 5.14 Integrating demand over time to get total units sold

When a game goes on sale the number of units sold ramp up rapidly and then drop off. A sales consulting firm estimates that the weekly sales as a function of the number of weeks is $D(w) = \frac{10w}{10w^2+1}$ *hundreds of thousands of units. Use an integral to estimate the number of weeks until 500,000 units are sold.*



The integral of the function $D(w)$ *is the total units sold so solve:*

$$5 = \int_0^n \frac{10w}{10w^2+1}\cdot dw$$

Remember that 500,000 is 5-hundred thousand. Notice that this needs u-substitution with $u = 10w^2 + 1$ *and* $\frac{1}{2}du = 10w$ *so:*

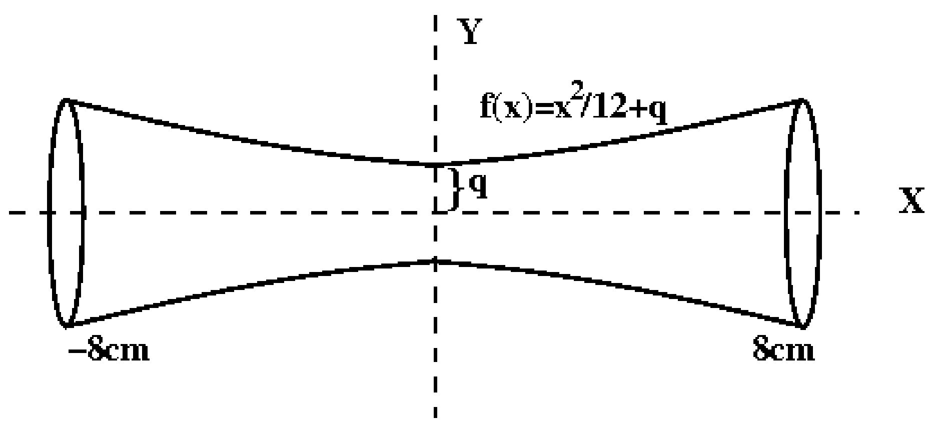
$$\begin{aligned}
\int \frac{10w}{10w^2+1}\cdot dw &= \frac{1}{2}\int \frac{1}{u}\cdot du \\
&= \frac{1}{2}ln(u) + C \\
&= \frac{1}{2}ln\left(10w^2+1\right) + C
\end{aligned}$$

Plugging the result of this indefinite integral into the original integral we get

$$
\begin{aligned}
5 &= \frac{1}{2} ln\left(10w^2+1\right)\Big|_0^n \\
5 &= \frac{1}{2} ln(10n^2+1) - 0 \\
10 &= ln(10n^2+1) \\
e^{10} &= e^{ln(10n^2+1)} \\
e^{10} &= 10n^2+1 \\
n^2 &= \frac{e^{10}-1}{10} \\
n &= \sqrt{\frac{e^0-1}{10}} \\
n &\cong 46.93
\end{aligned}
$$

So the sales goal of 500,000 units should be met around week 47.

In the next example we show how to use integrals to total up cross sectional areas in order to obtain a volume.



Example 5.15 Sizing a novelty beer mug

Problem: *The diagram above shows the design for a novelty beer glass to be made out of high impact plastic. The cross section of the mug is circular but the radius of the cross section is given by the formula* $r(x) = q + \frac{x^2}{12}$. *The mug's design goes from* $x = -8$*cm to* $x = +8$*cm. The only number we are allowed to change is* q*, in centimeters, which controls how thick the glass is at its stem. Find the value of* q *that yields a 1 liter glass.*

Solution: *First notice that the volume for positive and negative* x *is the same, so we only need to solve for the volume from* $x = 0$ *to* $x = 8$ *and make it half a liter (500cc's). The rate at which the volume changes, as we go down the* x*-axis from* $x = 0$ *to* $x = 8$ *is the area of the cross section. Circles have an area of* $A = \pi r^2$ *so we can find the volume by integrating the areas of the circles that form cross sections of the mug.*

We are given that the radius is $r(x) = q + \frac{x^2}{12}$ *and so we want* q *so that:*

$$\begin{aligned}
500 &= \int_0^8 \pi r(x)^2 \cdot dx \\
&= \int_0^8 \pi \left(\frac{x^2}{12} + q\right)^2 \cdot dx \\
&= \pi \int_0^8 \left(\frac{x^4}{144} + 2\frac{x^2}{12}q + q^2\right) dx \\
&= \pi \left(\frac{x^5}{720} + \frac{qx^3}{18} + q^2 x\right)\Big|_0^8 \\
&= \pi \left(\frac{32768}{720} + \frac{512q}{18} + 8q^2 - 0 - 0 - 0\right) \\
0 &= \frac{1024}{45}\pi - 500 + \frac{256\pi}{9}q + 8\pi q^2
\end{aligned}$$

We get a quadratic equation with variable q. *Since we are finding a physical length, it is appropriate to move to decimal approximations:*

$$-428 + 89.4q + 25.1q^2 = 0$$

Applying the quadratic formula (and ignoring the negative root) we get:

$$q = \frac{-89.4 + \sqrt{89.4^2 - 4(-428)(25.1)}}{50.2} \cong 2.72cm$$

Which means the glass will have a waist with a radius of 2.72cm or a diameter of 5.44cm. This is a fairly reasonable design, maybe a little thin in the middle. Of course the plastic can be made thicker, leaving the volume for beer the same, but this might increase materials costs for manufacturing too much. We will revisit this problem in the exercises.

A type of curve with the same sort of shape as that in Example 5.14 is a polynomial times e^{-x}. Since increasing the degree of the polynomials permits a statistician or modeler to select the degree of accuracy possible for the model, we include the integral formula for this type of curve.

Fact 5.6 *Suppose that* $f(x) = p(x)e^{-x}$ *where* $p(x)$ *is a polynomial. Then*

$$\int f(x) \cdot dx = -\left(p(x) + p'(x) + p''(x) + \ldots\right) e^{-x} + C$$

The terms between the large parenthesis are the sum of all *the non-zero derivatives of* $p(x)$*; don't worry, the number of derivatives is always finite if* $p(x)$ *is polynomial.*

Example 5.16 Demonstrating Fact 5.6

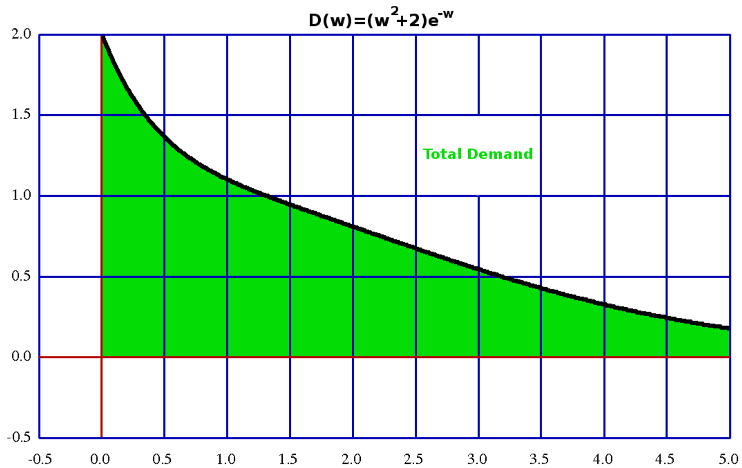
- $\int xe^{-x} = -(x+1)e^{-x} + C.$
- $\int x^2e^{-x} = -\left(x^2 + 2x + 2\right) e^{-x} + C.$
- $\int x^3e^{-x} = -\left(x^3 + 3x^2 + 6x + 6\right) e^{-x} + C.$
- $\int \left(x^2 + x + 1\right) e^{-x} = -\left((x^2 + x + 1) + (2x + 1) + (2)\right) e^{-x} + C = -\left(x^2 + 3x + 4\right) e^{-x} + C.$

Example 5.17 Another demand curve

Problem: *Suppose that demand, in millions of units, for a hair clip with a logo from a recent movie has a weekly demand curve $D(w)$ given by*

$$D(w) = (w^2 + 2)e^{-w}$$

Find the total production needed to satisfy demand for the first ten weeks.



Solution: *Integrate from $w = 0$ to $w = 10$, using Fact 5.6. The green area in the diagram above represents the total demand over time.*

$$\begin{aligned}
\int_0^{10} (w^2+2)e^{-w} \cdot dw &= -\left(w^2+2+2w+2\right)e^{-w}\Big|_0^{10} \\
&= -\left(w^2+2w+4\right)e^{-w}\Big|_0^{10} \\
&= -\left(124e^{-10} - 4e^0\right) \\
&= 4 - 124e^{-10} \\
&= \cong 3.994
\end{aligned}$$

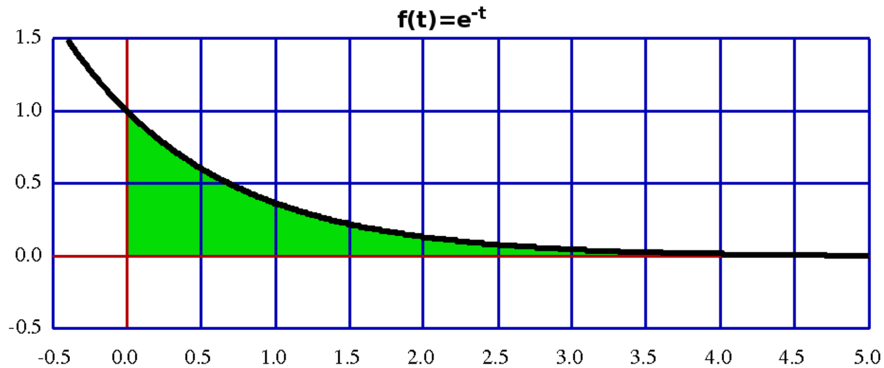
So the total demand in the first 10 weeks is 3,994,000 hair clips, assuming the technique used to estimate the demand curve is good.

Probability is an important application area for integrals. If you take a course in statistics you may see many of these applications. We include a single application, to demonstrate the potential.

Example 5.18 Lightbulb Failure

The probability an old-style incandescent light bulb will fail has an odd property - if the bulb is still burning then its remaining lifetime does not depend much on how long the light bulb has been burning. A new light bulb has roughly the same chance to burn out in the next few hours as one that's been in use for ten years (as long as the old bulb is still burning). This means that, ignoring some constants that will just make the problem harder, that the failure rate of light bulbs burning after t years is $f(t) = e^{-t}$. It turns out that the fraction of lighbulbs that fail from time $t = 0$ to time $t = H$ is the integral of the failure rate.

Problem: *If we are studying a large number of lightbulbs, estimate the time requred for half of them to fail.*



Solution: *Integrate the failure rate from $t = 0$ to $t = H$ where the fraction of failed bulbs is 1/2.*

$$\begin{aligned} 0.5 &= \int_0^H e^{-t} \cdot dt \\ &= -e^{-t}\Big|_0^H \\ &= -e^{-H} - (-e^0) \\ &= 1 - e^{-H} \\ e^{-H} &= 0.5 \\ -H &= ln(0.5) \\ H &= -ln(0.5) \\ H &= ln(2) \cong 0.693 \ years \end{aligned}$$

This means that the time at which close to half of a group of bulbs will fail is a little over 8 months. Notice we integrated the function with Fact 5.6.

Exercises

Exercise 5.21 *Do the following indefinite integrals, using Fact 5.6.*

a) $f(x) = \int x^3 \cdot e^{-x} \cdot dx$. **b)** $f(x) = \int x^4 \cdot e^{-x} \cdot dx$. **c)** $f(x) = \int \left(x^2 + 2x + 4\right) \cdot e^{-x} \cdot dx$.

d) $f(x) = \int \left(x^2 - 3x + 3\right) \cdot e^{-x} \cdot dx$. **e)** $f(x) = \int \left(x^3 + x^2 + x + 1\right) \cdot e^{-x} \cdot dx$.

f) $f(x) = \int \left(x^4 + 5x^2 + 1\right) \cdot e^{-x} \cdot dx$.

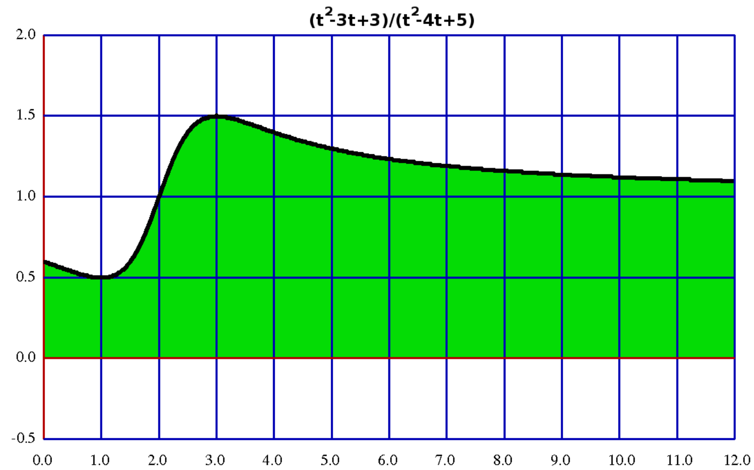
Exercise 5.22 *For each of the following functions, find a constant B so that $\int_0^B f(x) \cdot dx$ is equal to the given value.*

a) $f(x) = x + 1$, total integral is 20. **b)** $f(x) = 2x + 3$, total integral is 60.

c) $f(x) = 3x^2 - 4x + 1$, total integral is 2. **d)** $f(x) = 3x^2 - 8x + 2$, total integral is 8.

e) $f(x) = \frac{x}{x^2+1}$, total integral is 4. **f)** $f(x) = \frac{x^2}{x^3+1}$, total integral is 2.

Exercise 5.23 *Suppose that profits from sales of a line of kitchen appliances is $f(t) = 1.1 + 0.6t$ in month t of a year as sales ramp up toward Christmas and year-end sales. Compute the total profits for the year.*



Exercise 5.24 *Suppose that profits from sales of garden tools and supplies is $f(t) = \frac{t^2-3t+3}{t^2-4t+5}$ in month t of a year. Note that the maximum profits are in April. Compute the total profits for the year. A graph of the situation is shown above. Hint: start with long division.*

Exercise 5.25 *Using the results in Example 5.14, find the total units sold in the first year (52 weeks).*

Exercise 5.26 *Using the results in Example 5.14, estimate the number of years needed to sell one million units. The answer is a little surprising.*

Exercise 5.27 *Suppose instead of going from $x = -8cm$ to $x = 8cm$ the beer glass in Example 5.15 went from $x = -6cm$ to $x = 6cm$. Then what would the value of q be?*

Exercise 5.28 *Suppose in Example 5.15 we want a 2 liter glass. What does q become?*

Exercise 5.29 *Suppose in Example 5.15 we used a profile of $f(x) = x^2/25 + q$ instead of $f(x) = x^2/12 + q$. What would the value of q be?*

Exercise 5.30 *Suppose in Example 5.15 we simply set $q = 3$; what is the volume of the resulting glass?*

Exercise 5.31 *If the failure rate of a given type of incandescent light bulb after t years is*

$$f(t) = 4 \cdot e^{-4t}$$

find the time for roughly half the bulbs to burn out.

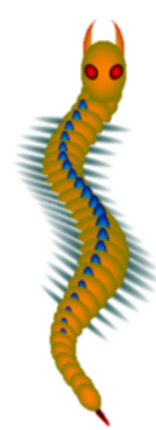
Exercise 5.32 *If the failure rate of a given type of incandescent light bulb after t years is*

$$f(t) = 6 \cdot e^{-6t}$$

find the time for half the bulbs to burn out.

Exercise 5.33 *Using Fact 5.6 and u-substitution, find a formula for*

$$\int p(x)e^x \cdot dx$$



Done with calculus!

Chapter 6

Systems of Linear Equations

A system of equations is a collection of two or more equations with a same set of unknowns. In solving a system of equations, we try to find values for each of the unknowns that will satisfy every equation in the system.
-Anon.

We will have a need for subtraction, and occasionally we will divide, but mostly you can describe linear equations as involving only addition and multiplication.
-Robert A. Beezer

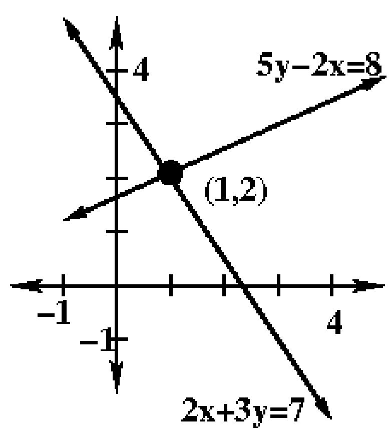
You don't hate linear algebra, you hate the class you're in. There's a difference. Linear Algebra is a beautiful subject.
Cyrus, from the "I Hate Linear Algebra" forum.

This section offers a treatment on setting up and solving linear systems of equations. In Chapter 1 we solved for the intersection of pairs of lines. This is the simplest version of the more complicated skill we will be building in this section. It would be good to review Section 1.2.1 before diving into this section. The section is also arithmetic intensive so drink your power tea, do your breathing exercises, or lay in a supply of your favorite snack, whatever it takes to do a whole lot of arithmetic accurately.

In Chapter 1 there was an easy picture to work with. When solving a system like:

$$\begin{aligned} 2y + 3x &= 7 \\ 5y - 2x &= 8 \end{aligned}$$

We could just find the point where the lines, corresponding to the equations, intersect. This is shown in the diagram at the right.



The answer $x = 1$, $y = 2$ that satisfies both the equations at the same time is available with a little algebra. In this section, we will introduce a different method, than the one in Chapter 1, which would be slightly harder if we applied it to two lines, like those above with two variables, but which is easier when there are three or more equations and variables.

Application: Expansion Packs of Cards

Suppose we have 300 darkside cards and 400 lightside cards that we want to put into 10-card expansion packs for a trading card game. We make solar packs, with two dark and eight light cards that sell for \$3.00 and nightfall packs that have five of each sort of card and sell for \$5.00. What is the best allocation of cards to packs?

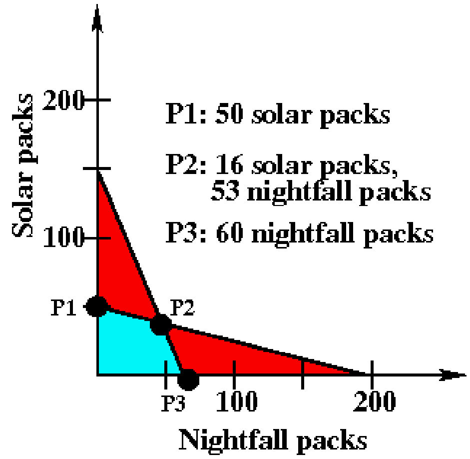
Let S be the number of solar packs and N be the number of nightfall packs. Then the fact we have 300 darkside cards tells us

$$2S + 5N = 300$$

while the fact that we have 400 lightside cards tells us that

$$8S + 5N = 400$$

If we graph these linear equations that describe the possible distribution of packs we get:



The red regions are areas where we don't have enough of one type of card. The blue regions are all the pairs (S,N) of packs we can make. Since unsold cards are a loss, we want solutions on the boundary of the blue and red regions. If we are on one of the lines that forms the boundary, revenue will change as we move along a line. This means that maximum revenue is at a place not only on the boundary of the blue region but where lines intersect. In other words, we only need to check P_1, P_2 and P_3. For P_2 we solve and round to the nearest whole numbers in the blue region getting $S = 16$, $N = 53$. P_1 is the "make only solar packs" solution which means 400/5=50 solar packs. P_3 is the "make only nightfall packs" solution which means 300/5=60 nightfall packs. Let's compare:

Point	Packs	Revenue
P_1	S=50, N=0	\$150.00
P_2	S=16, N=53	\$313.00
P_3	S=0, N=60	\$300.00

An so P_2 yields the maximum revenue. Notice that this is an optimization problem, but a very different sort from the calculus based optimization in Chapter 4.

How to Set Up a Linear Equation

Consider the following example. You inherited $55,000 and invested part of it in a tax-free savings account, part in government bonds, and part in a mutual fund. After one year, you received a total of $2,950 in simple interest from the three investments. The TFSA paid 5% annually, the bonds paid 6% annually, and the mutual fund paid 7% annually. You invested $9,500 more in the mutual funds than in the TFSA. Find the amount you invested in each.
It is clear that there are three things we are looking for:

1. The amount of money you invested in the TFSA

2. The amount of money you invested in bonds

3. The amount of money you invested in the mutual fund

The first helpful thing to do is assign a variable to each item we are looking for, since rewriting them over and over seems inefficient. Let's call the amount of money in the TFSA x, the amount of money in the bonds y, and the amount in the mutual funds z. From our example, we know three things for sure. The first is we know how much total money was invested, and we can write that in the form:

$$x + y + z = 55000$$

We also know how much money was returned from the simple interest. You got $2,950, and we know how much what the percentage was on each investment. So we can write that in the form:

$$0.05x + 0.06y + 0.07z = 2950$$

We have one last piece of information, that the amount in the mutual funds minus the amount in the TFSA is $9,500, which we can write as:

$$x - z = 9500$$

We now have the following three equations, which create a system:

$$\begin{aligned} x + y + z &= 55000 \\ 0.05x + 0.06y + 0.07z &= 2950 \\ x - z &= 9500 \end{aligned}$$

The process we just went through is called *setting up a linear system.* We still need to teach you to *solve* such systems.

Why are they called "linear" equations?

An equation like $2y + 3x = 7$ can be solved to yield

$$y = -\frac{3}{2}x + \frac{7}{2},$$

clearly the point-slope form of a line. An equation line

$$0.05x + 0.06y + 0.07z = 2950,$$

however, is *not* something you can solve to get a line. An equation is **linear** if the variables are all to the first power and not multiplied by one another.

If we have a three variable equation line $x + y + z = 3$ then it does have a graph, but it is not a line. Instead it is a **plane** (flat surface) in three dimensions. An equation like:

$$a + b + c + d = 12$$

has a graph that is a solid (three dimensional) object in a four dimensional space. This object is called a **three dimensional hyperplane**. This is a little hard to picture, and so we use the abstraction of linear equations to work with it.

At this point you may be wondering how objects in four dimensions are relevant to the real world in general, or business in particular. Consider the problem of allocating airplanes, food, fuel, pilots, cabin staff, maintenance personnel, replacement parts, supplies like lube and cleaning supplies, drinks (first class and main cabin, they're *very* different, etc. etc. for an airline. This can involve *hundreds* of linear equations in hundreds of variables. This means a problem whose geometry involves a space with hundreds of dimensions comes up every day. The formalisms in this chapter, programmed into computers with a little sophistication, can solve these problems. If you're interested, look up an article on *linear programming.*

A final word: a plane is the 3-dimensional generalization of a line. A 3-hyperplane in 4 dimensional space is the 4-dimensional generalization of a line or plane. Since these objects are generalized lines, we call the whole area of study "linear" algebra.

A Crucial Fact

When solving linear systems, if you have n variables to solve for, you need at least n equations to do the job. So if we have 2 variables, we need at least 2 equations, 3 variables, at least 3 equations, and so on. When setting up a linear system, make sure all your variables are accounted for, and that you have a sufficient number of equations to solve them. If you don't have enough equations then you cannot get a unique solution (there may be many). If you have too many equations then either some of them contain redundant information *or* it may be impossible to solve the system. This often happens when solving real problems, because someone wants something that really isn't possible.

A Few Things to Remember

Here are a few rules to know about the manipulation of the pieces of a system of linear equations, including a review of elimination . Let us consider the following example:

$$2x - 3y + 4z = 5 \tag{6.1}$$
$$x + y + z = 1 \tag{6.2}$$
$$4x + y - 2z = -3 \tag{6.3}$$

Here are three distinct linear equations. When we set up a linear system, solving it is the same as finding where the planes that the equations represent intersect, if they do so at all. These equations can be rearranged, according to whatever preference you may have, and it will still be the same linear system. Multiplying any equation by a constant (and remembering that we have to multiply every piece of the equation on both sides of the equals sign by that constant) also does not change the linear system. For example, if we were to switch equations 2 and 3, and multiply equation (1) by 2, the result would be:

$$4x - 6y + 8z = 10 \tag{6.4}$$
$$4x + y - 2z = -3 \tag{6.5}$$
$$x + y + z = 1 \tag{6.6}$$

The solution to this system is the same as our original system. Thus, we can apply any manipulation that involves multiplying by a constant, or shifting the position of an equation, but the result is still the same system. We can also add (or subtract) two equations without changing the solution of the system. For example, if we now added equation 5 to equation 6, the result would be:

$$4x - 6y + 8z = 10 \tag{6.7}$$
$$4x + y - 2z = -3 \tag{6.8}$$
$$5x + 2y - z = -2 \tag{6.9}$$

Although this system now looks fairly different from the one with started with originally, they share the same unique solution. Keeping this idea in hand, we move to the technique of elimination. As a quick aside worth mentioning, we can multiply an equation by any real number, not just integers. So the numbers $\frac{1}{3}, \sqrt{5}, 12.7$, etc., are all perfectly acceptable constants. However, don't make the mistake of multiplying by variables, since that will definitely change the system of equations you are working with and inevitably give you the wrong answer.

Elimination to Solve Linear Equations

Elimination is the tool we are going to use to help us solve our linear system problems. We'll use the first example and explain what is happening step by step:

$$2x - 3y + 4z = 5(1)$$
$$x + y + z = 1(2)$$
$$4x + y - 2z = -3(3)$$

Elimination, like the name suggests, is about isolating and solving for a variable in an equation by removing (eliminating) all the other variables. It involves multiplying equations by constants and adding them to other equations. It is an extremely useful technique, since all of our operations on these equations in the system won't change the answer. Let's consider the first equation (1), and see what happens when we remove the z variable. The first thing we have to look at is the coefficient of z in (1), in this case it is 4. Now we can use either equation (2) or (3). Let's choose (2), and perform the following operations. First, multiply (2) by 4, giving us 4x + 4y +

4z = 4, then subtract the new equation from (1). If we write this operation down, it would look like a subtraction with regular numbers.

$$\begin{array}{r} 2x - 3y + 4z = 5 \\ - \underline{4x + 4y + 4z = 4} \\ -2x - 7y = 1 \end{array}$$

A key idea to remember is that even though we performed these operations on (2) and then combined it with (1), the original (2) does not change. This bears repeating. The equation we choose to manipulate and add (or subtract) to another equation remains unchanged in the system. So the following operations yield:

$$\begin{aligned} -2x - 7y &= 1 \\ x + y + z &= 1 \\ 4x + y - 2z &= -3 \end{aligned}$$

We can continue in this vein and eliminate z from (3). We first multiply (2) by -2, giving us -2x -2y -2z = -2, to make the coefficients of z in both equations equal. Then we subtract our new equation from (3).

$$\begin{array}{r} 4x + y - 2z = -3 \\ - \underline{-2x - 2y - 2z = -2} \\ 6x + 3y = -1 \end{array}$$

Our system now looks like:

$$\begin{aligned} -2x - 7y &= 1 \\ x + y + z &= 1 \\ 6x + 3y &= -1 \end{aligned}$$

We could continue to eliminate variables from each equation until we solved for each one, and in the next section we will learn the basic procedure to solve the system.

Procedures for Solving Linear Systems of Equations

Following this procedure will efficiently solve any system of linear equations. The best way to learn this procedure is by practicing it several times with different systems of linear equations. The first few will be rather time consuming, but once you have the process down it becomes significantly easier and quicker. Like most things, repetition is the key to success here. These steps are guaranteed to solve the linear system and later on, we will go through some examples to demonstrate possibilities and some more subtle aspects solving linear systems. Listed below are the steps:

1. Start with the last (bottom) equation of the linear system.
2. Consider the variable that is furthest to the right. If the coefficient of that variable is 1, do nothing. If it is not one, multiply every variable and constant in that equation by the number that will change the coefficient of the variable you started with to 1.
3. Eliminate the variable from all the equations above the last equation by multiplying each equation by the appropriate constants and subtracting them.

4. Now move up to the equation that is right above the equation we were working with, and consider the furthest variable to the right in our new equation. Repeat steps 2 and 3.
5. Continue until the first equation has the first variable remaining, the second equation has the first 2 variables remaining, and so on, with the nth (last) equation has n variables remaining.
6. Now consider the first variable in the first equation. If the coefficient is not 1, multiply the first equation by whatever number will make the coefficient 1. We use a similar method to the way we removed variables from equations above the equation and variable we were working with. Now we work down from the top equation and remove all the variables in the equations beneath the first equation.
7. Move to the second equation, there should only be the second variable remaining. Eliminate the second variable from all the equations beneath it.
8. Eliminate all variables from the equations until each equation has only one variable remaining. This is the solved system.

These steps are guaranteed to solve any system of linear equations (or demonstrate they cannot be solved), however we can replace steps 6-8 with the following:

1. Consider the first variable in the first equation. If the coefficient is not 1, multiply the first equation by whatever number will make the coefficient 1. When the coefficient is 1, then we know that variable is equal to whatever constant is on the other side of the equals sign.
2. When the value of the first variable is known, plug it into the second equation, and solve for the second variable.
3. Plug the values of the first and second variables into the third equation, and solve for the third variable.
4. Continue until all the variables are solved for.

The steps in the original process or the ones offered in the alternative will both solve the system correctly, it just becomes a matter of personal preference. Use the methods that makes a given set of equations easier to solve.

Two Examples

Here are two examples to clearly illustrate what is happening when we solve linear systems.
Example 1

$$\begin{aligned} x - 2y + 3z &= 7 \\ -3x + 2y - 2z &= -10 \\ 2x + y + z &= 4 \end{aligned}$$

Let's follow the procedure outlined in the previous section. Since the coefficient of z is 1 in the bottom equation, we don't need to multiply by anything. To eliminate the z variable in the middle equation, multiply the bottom equation by 2 and add it to middle equation. For the top equation, multiply the bottom equation by 3 and subtract it from the first equation. This yields:

$$\begin{aligned} -5x - 5y &= -5 \\ x + 4y &= -2 \\ 2x + y + z &= 4 \end{aligned}$$

Before we focus on the middle equation, we can make our lives easier by noticing that every element in the first equation has a coefficient of 5. If we multiply the top equation by $\frac{1}{5}$, we get the follow equation, which will be easier to work with.

$$\begin{aligned} -x - y &= -1 \\ x + 4y &= -2 \\ 2x + y + z &= 4 \end{aligned}$$

Now we focus on the middle equation, and the second variable. We want to eliminate the second variable in the first equation, so we multiply the first equation by 4 and subtract it from the second equation.

$$\begin{aligned} -3x &= -6 \\ x + 4y &= -2 \\ 2x + y + z &= 4 \end{aligned}$$

Using a similar trick from before, we can change the first equation by multiplying it by $\frac{1}{3}$, which will give us the equation x = 2. Then to eliminate the first variable from the second equation, we subtract the first equation from the second equation. To eliminate it from the third equation, we multiply the first equation by 2 and subtract it from the third equation, yielding:

$$\begin{aligned} x &= 2 \\ 4y &= -4 \\ y + z &= 0 \end{aligned}$$

Before the last step, we multiply the second equation by $\frac{1}{4}$ to obtain y = -1. To eliminate the second variable from the third equation, we subtract the second equation from the third equation, giving us the solution to the system:

$$\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 1 \end{aligned}$$

Example 2

Unfortunately, unlike the first example most linear systems don't result in clean, whole number solutions. The process is still the same, as the following system of equations will demonstrate.

$$\begin{aligned} 5x + 4y + 3z &= 8 \\ 2x + 7y + 5z &= 5 \\ 4x + 4y + 2z &= 4 \end{aligned}$$

The coefficient of z is 2 in the bottom equation,so we need to multiply the whole equation by $\frac{1}{2}$, giving us:

$$\begin{aligned} 5x + 4y + 3z &= 8 \\ 2x + 7y + 5z &= 5 \\ 2x + 2y + z &= 2 \end{aligned}$$

To eliminate the z variable in the middle equation, multiply the bottom equation by 5 and subtract it from the middle equation. For the top equation, multiply the bottom equation by 3 and subtract it from the first equation. This yields:

$$\begin{aligned} -x - 2y &= 2 \\ -8x - 3y &= -5 \\ 2x + y + z &= 2 \end{aligned}$$

Now we focus on the middle equation, and the second variable. We multiply the second equation by $-\frac{1}{3}$, in order to turn the coefficient of the second variable into 1:

$$\begin{aligned} -x - 2y &= 2 \\ \frac{8}{3}x + y &= \frac{5}{3} \\ 2x + y + z &= 2 \end{aligned}$$

To eliminate the second variable from the first equation, multiply the second equation by 2 and add it to the first.

$$\begin{aligned} \frac{13}{3}x &= \frac{16}{3} \\ \frac{8}{3}x + y &= \frac{5}{3} \\ 2x + y + z &= 2 \end{aligned}$$

We want the coefficient of the first variable in the first equation to be 1, so we multiply by $\frac{3}{13}$, obtaining:

$$\begin{aligned} x &= \frac{16}{13} \\ \frac{8}{3}x + y &= \frac{5}{3} \\ 2x + y + z &= 2 \end{aligned}$$

Now we want to eliminate the first variable from the second and third equations. To eliminate it from the second equation, multiply the first equation by $\frac{8}{3}$ and subtract it from the second equation. To eliminate it from the third equation, multiply the first equation by 2 and subtract it from the third equation:

$$\begin{aligned} x &= \frac{16}{13} \\ y &= -\frac{21}{13} \\ 2y + z &= -\frac{6}{13} \end{aligned}$$

To solve for the last variable, multiply the second equation by two and subtract it from the third, which gives the solution:

$$x = \frac{16}{13}$$
$$y = -\frac{21}{13}$$
$$z = \frac{36}{13}$$

Two Special Cases

Not every system of equations has a unique solution. When solving a linear system keep an eye out for two possibilities: the system has an infinite number of solutions, or no solution at all. The case where there are an infinite number of solutions happens when one (or more) of the equations is made completely of redundant information. If you're trying to set up a linear system and accidentally use the same information twice in exactly the same way, you may create a linear system with an infinite number of solutions. In some situations, this is natural, and there is some freedom in the system that lets you choose the value of one (or more) of the variables.

The case where there is no solution has a simple geometric explanation: two of the lines, planes, or hyperplanes that correspond to your system of equations are parallel to one another. Since they have to intersect to yield a solution, there isn't one. If you want to think of this in terms of the problem you're solving, two of the pieces of information you're using to set up the linear system conflict with one another in some fundamental way.

Infinitely Many Solutions

A system has infinite solutions if, as a result of elimination, one of the equations becomes something of the form, 0x = 0. Alternatively, 0y = 0, or 0z = 0, or any combination such as 0x + 0y = 0, will give the same answer of infinite solutions; the point being that the coefficient of all the variables present in the equation are zero, and that they equal zero. Consider the following example:

$$\begin{aligned} 2x + 2y + 3z &= 10 \\ 3x - 2y + 2z &= 10 \\ -2x + 3y - 0.5z &= -5 \end{aligned}$$

Let's go through the process of solving the system and see what we get. We start with the variable furthest to the right, z and make its coefficient 1 by multiplying the bottom equation by -2:

$$\begin{aligned} 2x + 2y + 3z &= 10 \\ 3x - 2y + 2z &= 10 \\ 4x - 6y + z &= 10 \end{aligned}$$

Multiply the last equation by 2 and subtract it from the second equation, and then multiply the last equation by 3 and subtract it from the first equation, giving:

$$\begin{aligned} -10x + 20y &= -20 \\ -5x + 10y &= -10 \\ 4x - 6y + z &= 10 \end{aligned}$$

Now let's divide the first equation by 20, and the second equation by 10, to make the coefficient of y 1 in both equations:

$$
\begin{aligned}
-0.5x + y &= -1 \\
-0.5x + y &= -1 \\
4x - 6y + z &= 10
\end{aligned}
$$

Now let's eliminate y from the first equation, by subtracting the second equation from the first:

$$
\begin{aligned}
0x + 0y &= 0 \\
-0.5x + y &= -1 \\
4x - 6y + z &= 10
\end{aligned}
$$

At this point, we can stop. The first equation is now of the form $0x + 0y = 0$ (or we could write it $0x = 0$, or $0y = 0$, it doesn't really matter), which means there is an infinite number of solutions. What it means to have an infinite number of solutions is that *any* value of x will work, since zero times anything will still give us zero. The choice of x would be completely dependent on the type of problem these equations were representing. There may be some problems where you are required to choose a value for x. Let a be the choice you make for x, such that $x = a$. Remember, a is just some number, which gives us the system:

$$
\begin{aligned}
x &= a \\
-0.5a + y &= -1 \\
4a - 6y + z &= 10
\end{aligned}
$$

Now we can solve for the other two variables. If we shift $-0.5a$ in the second equation to the other side of the equals sign, we have the value of y. We can then plug that value of y into the third equation, and solve for z. Make sure you try those steps and it should give us the solution with an arbitrary number for x:

$$
\begin{aligned}
x &= a \\
y &= -1 + 0.5a \\
z &= 4 - a
\end{aligned}
$$

Hopefully, you noticed one very important thing about the nature of the equations. Here is the punchline for having an infinite number of solutions: whether the system starts out this way or it is achieved by steps of elimination, if one equation is exactly the same as another, **or** one equation is an exact multiple of another equation in the system, then that system has an infinite number of solutions.

No Solution

The other possibility is that a system has no solutions. It has no solutions if, through elimination we get one of the equations to be something of the form, $0x = c$, where c is some constant. Consider the following system:

$$
\begin{aligned}
2x + 2y + 3z &= 10 \\
3x - 2y + 2z &= 20 \\
-2x + 3y - 0.5z &= -5
\end{aligned}
$$

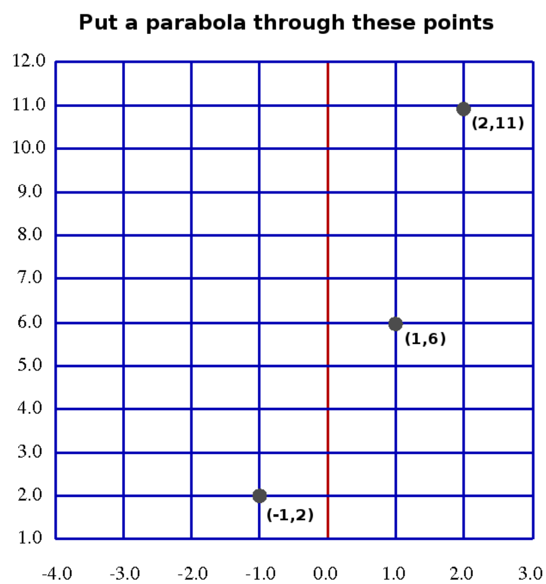
Notice that this is almost the same system from the example with infinite solutions, with the exception of the constant value in the second equation. If we repeat the same steps we went through in the previous example (make sure to try this on your own), we end up with the following system:

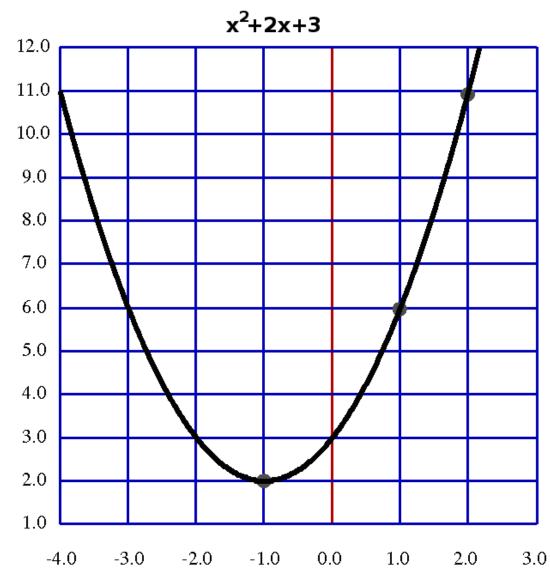
$$0x = 1$$
$$0.5x + y = 0$$
$$4x - 6y + z = 10$$

Clearly, there can be no solution, since zero times anything will not equal 1. Learn the difference between infinite solutions and no solutions: infinite solutions occur when we have zero times a variable equal to zero, which is why any value for that variable will work. No solutions occur when we have zero times a variable equal to some number, which is impossible. A system which has no solution is called *inconsistent*.

Application: Get The Parabola You Want

One thing we can do with linear equations is to fit a parabola to three points in the plane (as long as they are not all on the same vertical line). Look at the pictures below:





Remember that the general equation of a parabola is

$$y = ax^2 + bx + c$$

so if we plug in the three points (-1,2), (1,6), and (2,11) we get three equations (one per point).

The equations are as follows.

Point:	Equation from plugging in:
(-1,2)	$a - b + c = 2$
(1,6)	$a + b + c = 6$
(2,11)	$4a + 2b + c = 11$

Solve: add the first two equations and we get $2a + 2c = 8$ (fourth equation). Add twice the first equation to the third and we get $6a + 3c = 15$ (fifth equation). Subtract three times the fourth equation from the fifth and we get $-3c = -9$ so $c = 3$. Plug $c = 3$ into the fourth equation and get $2a + 6 = 8$ so $2a = 2$ and $a = 1$. Plus a and c into the first equation and get $1 - b + 3 = 2$ so $-b = -2$ and $b = 2$.

This means the equation in the left picture, $y = x^2 + 2x + 3$ is correct.

Exercises

Exercise 6.1 *Solve the following systems of equations using elimination:*

a) $x + y = 2,\ 2x - y = 0$

b) $-x + 2y = 10,\ 0.5x + 2y = -3$

c) $4x + 2y = 8,\ 0.5y = 2 - x$

d) $y - 7 = 3x + 2,\ x - 4y = 2x - y + 3$

e) $x + y + z = 2,\ -x + y - z = 1,\ 2x - 3y + 4z = 0$

f) $x - y - z = 3,\ 2x + 2y + 2z = -1,\ 3x - 3y - 3z = 15$

g) $x - 3y + 3z = -4,\ 2x + 3y - z = 15,\ 4x - 3y - z = 19$

h) $-10x + 2y + z = -8,\ 2x - y + 3z = 3,\ 100x + 600y - 300z = -200$

i) $x - y + z = -1,\ -x - y + 7z = 0,\ 2x + 4y - 3z + 5 = x + 5y - 4z$

j) $2x - 10y + 13z = -8y + 12z - 2,\ y = 2x - 4z + 10,\ 10x + 5y + 7z = 8x + 7y + 6z - 3$

Exercise 6.2 *Solve the first example:*

$$\begin{aligned} x + y + z &= 55000 \\ 0.05x + 0.06y + 0.07z &= 2950 \\ x - z &= 9500 \end{aligned}$$

Exercise 6.3 *Solve the following systems. Remember to use the step by step process, it will make solving larger systems like these much easier:*

a)

$$\begin{aligned} a + b + c + d &= 2 \\ 2a - b + 2c - d &= -1 \\ 3a - 3b - 3c + 3d &= 0 \\ -6a + 4b + 10c - 2d &= 4 \end{aligned}$$

b)

$$\begin{aligned} 4a - 3b - 2c + 5d &= 3 \\ -5a + b - c - 6d &= -9 \\ -2a - 6b + 8c + 4d &= 5 \\ -3a + 10b + c + d &= 3 \end{aligned}$$

c)

$$\begin{aligned} 4a - b - c - d &= -6 \\ 2a - 4b + c + 3d &= 0 \\ -a - 3b - 4c + 6d &= 0 \\ -3a + 4b + 5c + 3d &= 3 \end{aligned}$$

d)

$$\begin{aligned} a - c &= 12 \\ b + d &= 7 \\ 5a + 2b - 3c + d &= 4 \\ -2a + 4c - d &= 3 \end{aligned}$$

Exercise 6.4 *For each of the following problems, find the value(s) that make the system a) have infinite solutions, and b) inconsistent.*

a) $2x + 2y = 1,\ 2x + 2y = C$

b) $Ax + By = 4,\ x + y = C$

c) $x + y + z = 2,\ Ax + By + Cz = 2,\ x + y + z = E$

d) $2x + 4y + 5z = 47$, $3x + 10y + 11z = 104$, $3x + 2y + 4z = A$

Exercise 6.5 *The cost to eat at a buffet in a restaurant is \$6.50 for children and \$12.50 for adults. On a certain day, 620 people enter the restaurant and the total money earned is \$5230. How many children and adults ate at the buffet?*

Exercise 6.6 *The sum of the digits of a two-digit number is 8. When the digits are reversed, the number is increased by 18. Find the number.*

Exercise 6.7 *Find the equation of the parabola that passes through the points (2, 8), (0, 4), and (1, 11).*

Exercise 6.8 *You own a pie store and placed two orders at a grocery store. The first order was for 13 bushels of apples and 4 boxes of pie crusts, and totalled \$487. The second order was for 6 bushels and 2 boxes, and totalled \$232. The store just sent the bills and they didn't list the per-item price. What was the cost of one bushel, of one box?*

Exercise 6.9 *A riverboat cruise took 2 hours to travel 120 miles in the direction of the river. The return trip against the river took four hours. What was the boat's speed and the speed of the river?*

Exercise 6.10 *The Gryphons scored a total of 102 points in a resounding victory over the Mustangs. They had a field goal percentage of 0.76 on 50 shots. How many shots were successful? How many successful shots were worth two points? Three points?*

Exercise 6.11 *The standard equation of a circle is $x^2 + y^2 + Ax + By + C = 0$. Find the equation of a circle that passes through the points (4,0), (-1,5), and (7,9).*

Exercise 6.12 *Your company has three acid solutions on hand: 20%, 35%, and 75% acid. It can mix all three to come up with a 100-gallons of a 32% acid solution. If it interchanges the amount of 20% solution with the amount of the 75% solution in the first mix, it can create a 100-gallon solution that is 49% acid. How much of the 20%, 35%, and 75% solutions did the company mix to create a 100-gallons of a 39% acid solution?*

Congratulations!

This is the end!

Glossary

The definitions of terms that are specific to mathematics, and so might not appear in a useful form in a standard dictionary, are gathered here.

The origin of the work **algebra** is the Arabic word "al-jabr" which means (roughly) "reunion". It is the science of reworking statements about equality so that they are more useful. Chapter 1 covers most of the basics of algebra.

Antiderivative is another word for **integral**. The antiderivative of $f(x)$ is a function $F(x)$ so that $F'(x) = f(x)$.

An **Asymptote** is a line that the graph of a function approaches arbitrarily close to. If such a line is horizontal we call the asymptote a **horizontal asymptote**; if it is vertical we have a **vertical asymptote**.

Bending a function $f(x)$ is the result of either scaling the function by a constant $c \cdot f(x)$ or scaling the function's argument by a constant $f(cx)$. See Section 1.4.

Bounded sequences A sequence is *bounded above* if you can find a constant that is larger than every member of the sequence. A sequence is *bounded below* if you can find a constant that is smaller than every member of the sequence. A sequence is *bounded* if it is both bounded above and bounded below.
q A **chord** of a curve is a line that joins two points on the curve.

A **constraint** is one of two or more equations in an optimization problem that limit the domain of the optimization problem. Typically a constraint is solved and used to eliminate a variable.

The **common denominator** of two fraction is the smallest quantity that is a multiple of the denominator of both fractions. You must find the common denominator in order to add fractions that do not already have the same denominator.

The **comparison test** for series has two parts. The first says that series of positive terms that are all smaller, term-by-term, than a series that has a limit must, itself have a limit. The second part says that a series of terms that are larger, term-by-term, than a series that diverges to infinity must also diverge to infinity.

Completing the square is an algorithm for solving a quadratic equation. See Algorithm 1.2 in Section 1.2.2.

The **composition** of f and g is the result of plugging $g(x)$ into $f(x)$ as if it were the variable. We denote the compstion by $f(g(x))$ or by $(f \circ g)(x)$.

A **compounding period** is the interval of time for which interest is paid. "Compounded annually" means that interest is paid once a year. Sometimes and interest rate is given as "annually compounded monthly". In this case the rate must be divided by the number of periods to get the rate per period. So 5% annual interest, compounded monthly, is a complicated way of saying that the monthly interest rate is 5/12% and that the compounding period is one month.

A function is **concave up** in a region if the graph in that region is curved so that the shape could hold water (opens upward). A function is **concave down** if it is curved in the opposite fashion, opening downward.

A **constant** is a letter used to represent a number value that we don't know, but the value of the constant never changes for a given problem.

A function $f(x)$ is **continuous** at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Informally, a function is continuous at a if, when drawing the graph of the function near a you don't need to lift your pencil. A function is continuous **on an interval [a,b]** if it is continuous at every point in the interval.

Convergence for sequences and series. A sequence converges if it has a limit. A series converges if its sequence of of partial sums converges.

A **critical point** for a function $f(x)$ consist of and point $(a, f(a))$ or an x value a such that:

1. $f'(a) = 0$,
2. $f'(a)$ does not exist, e.g. because of a divide-by-zero.
3. The domain of $f'(x)$ or $f(x)$ ends at a, e.g. at zero for $f(x) = \sqrt{x}$.

A function is **decreasing** on an range $a \leq x \leq b$ if for all $a \leq c \leq d \leq b$ we have that $f(c) > f(d)$.

Decreasing sequences A sequence is decreasing if its terms grow smaller (if $a_n > a_{n+1}$ for all n).

The **degree** of a polynomial is the highest power in any term of the polynomial. The polynomial $1 + 2x + x^3$ had degree 3.

A **demand curve** tells us how many units of a commodity consumers will be willing to buy at a given price.

The **denominator** of a fraction is the part on the bottom. It says what the units of the fraction are, e.g. "thirds".

The **derivative** of a function is the rate at which the function is changing. It is also called the **marginal** of a function. it is denoted $f'(x)$ for a function $f(x)$. See Chapter 3.

The **discriminant** of a quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$. This quantity is used to determine how many roots or solutions a quadratic equation has.

The **domain** of a a function is the set of real numbers for which the function can be computed.

Elimination is the technique used to solve systems of linear equations by removing variables. The process involves removing variables from equations until there is one variable left in each equation in the system.

Equations are statements that assert equality between two statements. The statements on opposite sides of the equals sign must have the same value, once all the variables and constants are accounted for. Equations are also used for mathematical identities which are true regardless of the value plugged into a given variable. For example, $(x - 1)^2 = (x - 1)(x - 1) = x^2 - 2x + 1$.

An **exponential function** is any function of the form

$$y = c^x$$

where c is a positive constant.

A **factor** of an expression is a quantity that divides evenly into the expression. Thus 2 is a factor of 8 ($8 \div 2 = 4$) and $(x - 1)$ is a factor of $x^2 - 3x + 2$ ($x^2 - 3x + 2 \div (x - 1) = (x - 2)$).

The **first derivative test** is a test to determine if an optima is a maxima or minima. It is performed by examining sign diagrams for the first derivative.

The **general term** of a sequence

$$\sum_{n=0}^{\infty} a_n$$

is a_n. The sequence of general terms

$$\{a_n\}_{n=0}^{\infty}$$

is sometimes used in understanding the behavior of a series.

A **geometer** is a person who studies geometry or, more generally, mathematics.

A **geometric series** is any series where the ratio r of adjacent terms is constant. Such series always have the form

$$\sum_{n=0}^{m} d \cdot r^n$$

is they are finite and

$$\sum_{n=0}^{\infty} d \cdot r^n$$

if they are infinite. A geometric series is determined by is initial term d, its ratio r, and the number of terms in the series.

The **geometric series formula** has two forms. The first is for finite geometric series:

$$d\frac{1-r^{k+1}}{1-r} = \sum_{n=0}^{k} d \cdot r^n$$

The second in for infinite geometric series:

$$\frac{d}{1-r} = \sum_{n=0}^{\infty} d \cdot r^n$$

This latter formula only works when $-1 < r < 1$.

The **high-lo game** is a technique for finding roots of a function. It is usually used when other methods fail, because it is a lot of work. See Section 1.5.1 for examples and an algorithmic statement of the game.

Higher order derivatives of a function are the result of taking the derivative multiple times. If you start with a function and then compute the derivative four times, the result if the fourth derivative.

A system of equations is **inconsistent** if it has no solutions. If the equations in the system were represented as straight lines, having no solutions is the same as saying, all of these lines do not intersect at the same point.

A function is **increasing** on an range $a \leq x \leq b$ if for all $a \leq c \leq d \leq b$ we have that $f(c) < f(d)$.

Increasing sequences. A sequence is increasing if its terms grow larger (if $a_n < a_{n+1}$ for all n).

An **integer** is a number without any fractional part. Thus, ...-3, -2, -1, 0, 1, 2, 3.. and so on are integers, whereas the number 1.1, or $\frac{2}{3}$ are not integers.

The **integral** of $f(x)$ is a function $F(x)$ so that $F'(x) = f(x)$.

The **integrand** is the thing being integrated. So, for example, in $\int x^2 \cdot dx$, the integrand is x^2.

The **intercept** of a line is the value where it intersects the y axis. It is also the point on the line with x-coordinate zero. A line that does not have an intercept must be vertical.

The rule **invert and multiply** is used to divide fractions. See Section 1.1.5.

Limit of a function from above. If $f(x)$ is a function and as $x \rightarrow a$, using only values $x > a$ the value $f(x) \rightarrow L$ then L is the limit of $f(x)$ from above.

Limit of a function from below. If $f(x)$ is a function and as $x \rightarrow a$, using only values $x < a$ the value $f(x) \rightarrow L$ then L is the limit of $f(x)$ from above.

Limit of a function. If $f(x)$ is a function and as $x \rightarrow a$ the value $f(x) \rightarrow L$ then L is the limit of $f(x)$ at $x = a$. This is the case if the limit exists - it may not. See: *Limit from above* and Limit from below. For a limit to exist these limits must exist and agree.

Limit of a sequence. If $S = \{a_1, a_2, a_3, \ldots\}$ is an infinite sequence then we say the number L is the limit of the sequence S if for every positive number ϵ we can find a number N (which can be different for different values of ϵ) so that all terms a_i, for which $i \geq N$, differ from L by less than ϵ.

Linear comes from the Latin word *linearis*, which means *createdbylines*. A function is **linear** if it is of the form $a\mathbf{x} + b$, where a and b are constants and $\mathbf{x}$ is the variable. The variable must have an exponent of 1, and linear equations can be drawn as a straight line on a graph.

A **Logarithm** is defined as follows. Suppose that a, b, and c are constants and that $b > 0$. Then $log_b(a) = c$if and only if $b^c = a$.

Logarithmic functions are written $y = log_b(x)$. Because a power of a positive number b must be positive, logarithm functions only exist when $x > 0$.

The **marginal cost** of an item is the amount the total cost of production changes if you make one more item. Marginal cost often depends on the number of items being made.

A global **maximum** for a function is the largest value it can take on. A local **maximum** is a value of $y = f(c)$ that is larger that all other nearby values - a *hill top*.

A global **minimum** for a function is the smallest value it can take on. A local **minimum** is a value of $y = f(c)$ that is smaller that all other nearby values - a *valley bottom*.

Monotone sequence. A sequence is said to be *monotone* if it is either increasing or decreasing.

If i% interest is accruing in a compounding period then the **multiplier** for that period is $1 + \frac{i}{100}$, the amount you multiply the current balance by to get the new balance.

The **numerator** of a fraction is the part on the top.

Optimization is the process of finding the best possible value a formula or process can produce. This may be a maximum, minimum, or tradeoff among several goals depending on the exact problem being solved.

An **optimum** (plural **optima**) is a name used to describe either maxima or minima pf a function.

The **order of operations** is a convention about which operations are performed first. Operations that are performed before others are said to have higher precedence. See section 1.1.2.

A **p-series** is any series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where p is a real number. These series converge when $p > 1$ and diverge when $p \leq 1$.

A **percentage** is a way of expressing a number as out of 100. It is either expressed with a % sign, as a fraction, or a decimal. Thus, $\frac{6}{100}$ is the same as 6%, is the same as 0.06. These are all the same way of saying, "six out of one hundred".

A **perfect square** is a quantity that is the square of some other quantity. For constants, perfect squares are numbers like 4 and 9. When working with quadratic equations perfect squares are expressions like $(x-3)^2 = x^2 - 6x + 9$.

The **point-slope** form of a line is $(y - y_0) = m(x - x_o)$ where m is the slope of the line and (x_0, y_0) is a point on the line.

A **polynomial** is a sum of constant multiples of non-negative whole-number powers of a variable. Examples include $2x + 1$, $x^2 + x + 1$, $x^3 + 2x^2 - 4x - 7$, and $x^7 + 3x - 1$.

A **polynomial function** is a polynomial used as the rule of a function as in $y = x^2 + x + 1$.

The **power rule** for derivatives says that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

The **precedence** of an operation is its relative importance with respect to other operations. More important operations are performed first. See Section 1.1.2.

The **product rule** for derivatives says that $(f(x) \times g(x))' = f'(x) \times g(x) + f(x) \times g'(x)$

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$ where a is required not to be zero.

The **quadratic formula** solves quadratic equations. If $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

See Section 1.2.2.

A **quantity** is a generic term for a piece of, or and entire, algebraic expression. Quantities can be simple constants like 3 or π or they can be complex like $3x + 2y - 1$. A quantity is something you are currently treating as a coherent object, even though you may break it up or change it on the very next step of the problem you're working.

The **quotient rule** for derivatives says that $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Radical is another word for root.

The **ratio** of a geometric series is the constant value obtained when adjacent terms of the series are divided. In the form

$$sum_{n=0}^{\infty} d \cdot r^n$$

the value r is the ratio. **Ratio** is also a term for the result of dividing two numbers. Four and two have a ratio of 2. A ratio may also be expressed in an archaic notation like 3:2 meaning there are three of the first quantity to two of the second. A room with sixty men and forty women has a 3:2 male-to-female ratio.

The **range** of a function is the set of values that can result from computing that function.

A **rational function** is a function in the form of a fraction whose numerator and denominator are polynomials.

The **reciprocal** of a number of quantity is the result of dividing one by that number or quantity. The reciprocal of 2 is thus $\frac{1}{2}$. If a number or quantity has a reciprocal, then it is also the thing you multiply the quantity by to obtain one: $2 \times \frac{1}{2} = 1$.

The **reciprocal rule** for derivatives says that $\left(\frac{1}{f(x)}\right)' = \frac{-f'(x)}{f(x)^2}$

A fraction is said to be in **reduced form** if it has no common factors in it numerator and denominator. See Section 1.1.5.

A number c is a **root** of an equation $f(x)$ if $f(c) = 0$.

The **second derivative test** is a test to determine if an optima is a maxima or minima. If a function $f(x)$ has an optima then $f''(c) > 0$ indicates a minimum while $f''(c) < 0$ indicates a maximum. If $f''(c) = 0$ then the test yields no information.

A **sequence** is a list of numbers. If the list is finite then we say the sequence is **finite**. If the list is infinite then we say the sequence is **infinite**.

If R is the series $a_1 + a_2 + a_3 + \cdots$ then **the sequence of partial sums** of R is the sequence

$$\{a_1, (a_1 + a_2), (a_1 + a_2 + a_3), (a_1 + a_2 + a_3 + a_4), \ldots\}$$

The nth term of the sequence of partial sums for a series R is the sum of the first n terms of R.

A **series** is a list of numbers that are to be added up in the order given. If the list is finite then we say the series is **finite**. If the list is infinite then we say the series is **infinite**.

The symbol $\sum$ means "add up". It can be used with finite or infinite series, e.g.:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \sum_{n=1}^{\infty} n$$

A **sign chart** is a table showing where some quantity is positive, negative, or neither. When made for the first derivative of a function they portray increasing and decreasing ranges. When made for the second derivative they portray regions that are concave up or concave down.

The **slope** of a line is the ratio of the increase of y to the increase of x on any segment of the line. If (x_0, y_0) and $x_1, y_1)$ are a pair of points on the line then the slope m may be computed using the formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

The **slope-intercept** form of a line is $y = mx + b$ where m is the slope and b is the intercept.

Sliding is a technique for speeding up algebra operations in which you need to multiply or divide both sides of an equation by something. It requires that both sides of an equality be in the form of fractions, which can always be done, for example by imposing a denominator of 1.

The **second derivative** of a function is the derivative of the first derivative. It is denoted $f''(x)$

The **solution** of a problem is the answer, generally when solving for variable.

A **supply curve** tells us how many units of a commodity manufacturers will offer for sale at a given price.

A **system of equations** is a set of n equations, with each equation containing m variables. The system can be solved if $m \leq n$. The solution of a system of linear equations can be thought of as finding the points where the lines representing each equation intersect.
Translating a function consists of shifting its graft left, right, up or down. See Section 1.4.

A **tangent line** is a line that touches a curve at a single point.

A **variable** is a letter used to represent a number value that we don't know, and the value of the variable may change within the scope of a given problem.

A **vertical line** is a set of points of the form (c, y) where c is a constant and y can take on any value. The formula for such a line is $x = c$, it has no intercept, and its slope is considered to be infinite or undefined.

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Appendix A

Solutions to selected exercises

Following are solutions to some of the exercises in the text. It is better to work the exercise and then chack the solution than to try to work toward the solution. The first method will let you learn more, on average. Also remember that some of these solutions may be wrong because the people that wrote the notes suffer from a bad case of being human. Bring potential errors to your instructor and get help.

A.1 Selected Answers from Chapter 1

Problem 1.1

a) $x = 3$.

c) Move everything with an x to one side, everything without an x to the other, such that $xy - 2x = 1$. Then use the distributive property to pull x out of the left side, giving, $x\,(y-2) = 1$. Divide both sides by $(y-2)$ to isolate x, giving. $x = \frac{1}{y-2}$.

e) Take the third root of both sides. Since, $\left(\sqrt[3]{y+1}\right)^3 = y+1$, and $\sqrt[3]{27} = 3$, we are left with $y + 1 = 3$. Solving for y by subtracting 1 from both sides, $y = 2$.

g) Use the slide rule, move $(x-3)$ to the left side so we are left with $(x-3)y = x+2$. Then distribute y on the left side, $xy - 3y = x + 2$. Since we are solving for x, move x to the left side and move $-3y$ to the right side, remembering to change the signs as they cross the equals sign, $xy - x = 3y + 2$. Pull x out of the left side, $x(y-1) = 3y+2$. Divide both sides by $(y-1)$ to solve for x, $x = \frac{3y+2}{y-1}$.

i) Use the slide rule to get $y - 1 = 2(x+1)$. Distribute 2 through the right side, $y - 1 = 2x + 2$. Get $2x$ alone by subtracting 2 from both sides, $y - 3 = 2x$. Divide both sides by 2, $x = \frac{y-3}{2}$

Problem 1.3

For x = 0, $\sqrt[3]{\frac{0+2}{0-2}} = \sqrt[3]{-1} = -1$

For x = 1, $\sqrt[3]{\frac{1+2}{1-2}} = \sqrt[3]{-3} = -1.442$

For x = -1, $\sqrt[3]{\frac{-1+2}{-1-2}} = \sqrt[3]{\frac{1}{-3}} = -0.693$

Problem 1.5

This problem becomes easier if we recognize that $\left(\sqrt{x-1}+2\right)\left(\sqrt{x-1}-2\right) = (x-1-4) = x-5$, using the difference of squares property. This leaves us with $\frac{x-5}{1-x}$. Then,
$x = 2 \rightarrow \frac{2-5}{1-2} = \frac{-3}{-1} = 3$
$x = 3 \rightarrow \frac{3-5}{1-3} = \frac{-2}{-2} = 1$
$x = 4 \rightarrow \frac{4-5}{1-4} = \frac{-1}{-3} = 0.333$

Problem 1.7

$3\sqrt{x^2+1}+7$. First, square the value of x. Then add 1 to that value. Now take the square root, and multiply by 3. Finally, add 7.

Problem 1.9

- **a)** $\frac{255}{40} = \frac{51}{8}$. Common factor is 5.
- **c)** $\frac{255}{27} = \frac{85}{9}$. Common factor is 3.
- **e)** $\frac{125}{625} = \frac{1}{5}$. Common factor is 125.
- **g)** $\frac{9y^2}{3xy} = \frac{3y}{x}$. Common factor is $3y$.
- **i)** $\frac{abc}{abd+abe} = \frac{c}{d+e}$. Change the bottom into $abd + abe = ab(d + e)$, then divide the top and bottom by ab

Problem 1.11

- **a)** $\frac{1}{3} \times \frac{5}{8} - \frac{1}{2} = \frac{5}{24} - \frac{1}{2} = \frac{5}{24} - \frac{12}{24} = \frac{-7}{24}$.
- **c)** $\frac{x}{3} \div 2y - \frac{2}{3}x = \frac{x}{6y} - \frac{2x}{3} = \frac{x-4xy}{6y} = \frac{x(1-4y)}{6y}$.
- **e)** $\frac{1}{n} \times 1n + 2 + \frac{2}{n+1} = 3 + \frac{2}{n+1} = \frac{3n+5}{n+1}$.
- **g)** $\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}} = \frac{\frac{5}{6}}{\frac{1}{6}} = \frac{5}{6} \times \frac{6}{1} = 5$.
- **i)** $\frac{(x+y)(x-y)}{\frac{1}{x}+2} = \frac{x^2-y^2}{\frac{2x+1}{x}} = \frac{x\left(x^2-y^2\right)}{2x+1}$.

Problem 1.13

$\frac{x^2+1}{2-y^2} = 1$. Use the slide rule to achieve $x^2 + 1 = 2 - y^2$, now solve for x and y
$x = \pm\sqrt{1-y^2}$ and $y = \pm\sqrt{1-x^2}$.

Problem 1.15

$2 + \sqrt{1+\frac{a}{b}}$

Problem 1.17

The common denominator will be 420.

Problem 1.19

First we start with the whole pie and call it 1. Then subtracting one quarter from 1, $1 - \frac{1}{4} = \frac{3}{4}$. Now we have to find out what one quarter of 3 quarters is, $\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$ and subtract that from $\frac{3}{4}$. So, $\frac{3}{4} - \frac{3}{16} = \frac{9}{16}$. Now find what one quarter of $\frac{9}{16}$ is and subtract it from $\frac{9}{16}$. Repeat this process as follows:

$$1 - \frac{1}{4} = \frac{3}{4}$$
$$\frac{3}{4} - \frac{3}{16} = \frac{9}{16}$$
$$\frac{9}{16} - \frac{9}{64} = \frac{27}{64}$$
$$\vdots$$
$$\frac{81}{256} - \frac{81}{1024} = \frac{243}{1024}$$

Thus, $\frac{81}{1024}$ is the smallest piece of pie handed out.

Problem 1.21

a) $y = 2x + 5$. slope = 2, intercept = 5.

c) $2x - 4y = 5$. slope = $\frac{1}{2}$, intercept = $\frac{-5}{4}$.

e) $(y - 5) = 3(x - 1)$. slope = 3, intercept = 2.

g) A line parallel to $y = -2x + 5$ through the point (-1,3). Use the point-slope formula, slope = -2, intercept = 1.

i) The line of slope 2 through the point (a,b). Use the point-slope formula, slope = 2, intercept = -2a + b.

Problem 1.23

a) Set $2x + 1 = -x + 10$. Solve for x, $3x = 9$ or $x = 3$. Plug the x value into either equation to get $y = 7$.

c) $y = 2x+1$ and $2y-4x = 3$. First you need to put the second equation in slope intercept form, $y = 2x+\frac{3}{2}$, then set the equations equal to each other. $2x + 1 = 2x + \frac{3}{2}$, clearly has no intersection, since these are parallel lines.

e) $x + y + 1 = 0$ and $x - y - 1 = 0$. Transform both equations into slope intercept form, set them equal to each other. $x = 0$, $y = -1$.

Problem 1.25

a) $x^2+4x+3 = (x + 3)(x + 1)$. **c)** $x^2+2x-8 = (x+4)(x-2)$. **e)** $9x^2-6x+1 = (3x-1)(3x-1) = (3x-1)^2$. **g)** $x^2 - 12 = (x - \sqrt{12})(x + \sqrt{12})$. **i)** $6x^2 + 13x + 6 = (3x + 2)(2x + 3)$.

Problem 1.27

a) $x^2 + x + 1 = 0$. $a = 1$, $b = 1$, $c = 1$. No solutions, $1^2 - 4(1)(1) < 0$.

c) $x^2 + 2x + 1 = 0$. $a = 1$, $b = 2$, $c = 1$ 1 solution, $x = 1$.

e) $2x^2 + 5x + 7$. $a = 2$, $b = 5$, $c = 7$. No solutions, $5^2 - 4(7)(2) < 0$

g) $x^2 = 2 \to x^2 - 2 = 0$. $a = 1$, $b = 0$, $c = -2$ Two solutions, $x = \pm\sqrt{2}$

i) $(x + 1)(x - 6) = 4 \to x^2 - 5x - 10 = 0$. $a = 1$, $b = -5$, $c = -10$. Two solutions, $x = \frac{5}{2} \pm \frac{\sqrt{65}}{2}$

Problem 1.31

There are infinite solutions, but here is how you would find 2 different point-slope forms. For the first form, choose $x = 1$, which gives us $y = -3$. You can use those two numbers to build the point-slope forms, the first being $y+3 = 2(x-1)$. Try for another value of x, get a value for y, and then you can build the second line.

Problem 1.33

Check the slopes of the 3 lines that constitute the triangle. If two slopes are negative reciprocals of each other, then it is a right triangle.
Slope between (1,3) and (2, -1), $\frac{3+1}{1-2} = -4$
Slope between (1,3) and $(\frac{-6}{5}, \frac{-7}{-5})$, $\frac{3+\frac{7}{5}}{1+\frac{6}{5}} = \frac{1}{8}$
Slope between (2,-1) and $(\frac{-6}{5}, \frac{-7}{-5})$, $\frac{2+\frac{7}{5}}{-1+\frac{6}{5}} = 2$
Since none of these slopes are negative reciprocals, this is not a right triangle.

Problem 1.35

Any solution that encloses an area of 9 will work, but here is a simple one: the two vertical lines $x = 0$ and $x = 3$ and the two horizontal lines $y = 0$ and $y = 3$.

Problem 1.37

Set the supply curve equal to the demand curve, such that $\frac{q}{25} = 1000 - \frac{q}{30}$ and solve for q. $q = 150,000$ and $p = 6,000$.

Problem 1.39

We can use the roots to build the quadratic. Since the parabola opens downward, we put a negative sign in front of everything to start, $-(x-7)(x+2) = -(x^2-5x-14) = -x^2+5x+14$. Then we can complete the square using Algorithm 1.2,

$$-\left(x^2 - \frac{5}{2}\right) + \frac{81}{4} = 0$$

Telling us our maximum is at $\frac{81}{4}$.

Problem 1.41

First, set $P(n) = 0 = 25000 + 850n - 10n^2$. Factor out a 10, giving us: $0 = 10(2500 + 85n - n^2)$. Then use Algorithm 1.2. to complete the square, yielding, $-10\left(n^2 - \frac{85}{2}\right)^2 + 43062.5 = 0$. This tells us our vertex is at $\left(\frac{85}{2}, 43062.50\right)$, telling us that the maximum occurs between 42 and 43 consoles. Both 42 and 43 consoles, by plugging back into the profit formula, are found to yield a profit of$43060.00, which is the maximum.

Problem 1.43

a) $\left(2^2\right)^3 = 64$.

c) $x^2 \times x^{3/4} \div x^{5/2} = x^{2+\frac{3}{4}-\frac{5}{2}} = x^{\frac{1}{4}}$.

e) $\frac{2+2^{-1}}{2^{-2}+2^{-3}} = \frac{2+\frac{1}{2}}{\frac{1}{4}+\frac{1}{8}} = \frac{\frac{5}{2}}{\frac{3}{8}} = \frac{20}{3}$.

Problem 1.45

a) $log(x+1) = 4 \rightarrow x+1 = 10^4 \rightarrow x = 9999$.

c) $log_2(x-2) = 3 \to x - 2 = 2^3 \to x = 10.$

e) $log_x(12) = 5 \to 12 = x^5 \to x = \sqrt[5]{12}.$

g) $log_{1/4}(x) = 2 \to x = (\frac{1}{4})^2 \to x = \frac{1}{16}.$

i) $Ln\left(\frac{x-1}{x+1}\right) = 1 \to \frac{x-1}{x+1} = e \to x - 1 = e(x+1) \to x = \frac{e+1}{1-e}.$

Problem 1.47

$$\frac{\sqrt{2x+5}-2}{\sqrt{2x+2}-3} \times \frac{\sqrt{2x+2}+3}{\sqrt{2x+2}+3}$$

.

$$= \frac{\sqrt{4x^2+14x+10} - 2\sqrt{2x+2} + 3\sqrt{2x+5} - 6}{2x-7}$$

Problem 1.49

$$\frac{x^2+1}{7-\sqrt{x}} \times \frac{7+\sqrt{x}}{7+\sqrt{x}} = \frac{\left(x^2+1\right)\left(7+\sqrt{x}\right)}{49-x} = \frac{7x^2 + x^{\frac{5}{2}} + 7 + \sqrt{x}}{49-x}$$

Problem 1.51

$log\left(\frac{a^3}{b^2}\right) = log\left(a^3\right) - log\left(b^2\right) = 3log\left(a\right) - 2log\left(b\right) = 3(1.5) - 2(2.25) = 4.5 - 4.5 = 0$

Problem 1.53

$1.15^n > 3 \to nlog(1.15) > log(3) \to n > \frac{log(3)}{log(1.15)} = 7.86,\ n = 8.$

Problem 1.55

Using the Compound Interest Application, we simply use the formula: $T = D(1+d)^n$, where T is the total after interest, D is the original amount invested, d is the interest rate invested at, at n is the number of times its compounded. In this case, we know $T = 1218.99$, $D = 1000$, $n = 10$, and all that is left to solve for is d. Using the formula, we get:

$$1218.99 = 1000(1+d)^{10}$$

The first step is dividing both sides by 1000,

$$1.21899 = (1+d)^{10}$$

, then we take the 10th root of both sides,

$$\sqrt[10]{1.21899} = 1 + d$$

$$1.01999963 - 1 = d$$

$$.01999963 = d$$

Solving for d, we can see the interest is almost 2%.

Problem 1.57

The first part of the question is fairly easy, just use the formula, $1000(1.05)^{10} = 1628.89$. To calculate compound interest on a monthly basis, we need to find what d is going to be. To do that, divide the yearly interest by

12 to get the monthly interest, so $d = .05 \div 12 = .00416667$. We also need to recalculate n, since there are 10 years and 12 months per year, now $n = 10 \times 12 = 120$ Now we can use the formula, $1000(1.00416667)^{120} = 1647.01$. We get a difference of $1647.01 - 1628.89 = 18.12$.

Problem 1.62

A a = 1, b = 1, c = 1

B a = -1, b = -1, c = 4

C a = -4, b = 2, c = 5

D a = $\frac{1}{2}$, b = 0, c = -1

Problem 1.64

If we complete the square of $x^2 - 4x$, we get $(x-2)^2 - 4$. Currently the vertex is at (2, -4), so we change -4 into -1 to get the vertex (2, -1), and our new function is $(x-2)^2 - 1$.

Problem 1.66

We are looking for a new function $f(x) = A3^x + b$. If we plug 0 into the function, we get $f(0) = -1 = A(3^0) + b = A + b$ and if we plug 2 into the function we get $f(2) = 6 = A(3^2) + b = 9A + b$. We can use these two equations to now solve for our two variables (See chapter 6 for a detailed explanation).

$$\begin{aligned} A + b &= -1 \\ -\ \underline{9A + b} &\underline{= 6} \\ -8A &= -7 \\ A &= \frac{7}{8} \end{aligned}$$

So we know $A = \frac{7}{8}$, now we plug that into the first equation to solve for b. $\frac{7}{8} + b = 1 \rightarrow b = \frac{-15}{8}$. Our equation is $f(x) = \frac{7}{8}(3^x) - \frac{15}{8}$.

Problem 1.67

a) $x^3 + x^2 - 10x + 8$. Try x = 1, $1^3 + 1^2 - 10(1) + 8 = 0$, so we know $(x-1)$ divides the polynomial.

$$\begin{array}{r|l} & x^2 + 2x - 8 \\ \hline x - 1 & x^3 + x^2 - 10x + 8 \\ & \underline{-x^3 + x^2} \\ & 2x^2 - 10x \\ & \underline{-2x^2 + 2x} \\ & -8x + 8 \\ & \underline{8x - 8} \\ & 0 \end{array}$$

We are left with $x^2 + 2x - 8$, which factors into $(x-2)(x+4)$, so $x^3 + x^2 - 10x + 8 = (x-1)(x-2)(x+4)$.

c) $x^3 - 2x^2 - x - 6$. Try x = 3. $27 - 18 - 3 - 6 = 0$, so we know $(x - 3)$ divides the polynomial.

$$\begin{array}{r|l} & x^2 \quad + x + 2 \\ \hline x-3 & x^3 - 2x^2 \quad - x - 6 \\ & -x^3 + 3x^2 \\ \hline & \quad x^2 \quad - x \\ & \quad -x^2 + 3x \\ \hline & \qquad 2x - 6 \\ & \qquad -2x + 6 \\ \hline & \qquad\quad 0 \end{array}$$

We are left with $x^2 + x + 2$, which does not factor, so our answer is $(x - 3)(x^2 + x + 2)$.

e) $4x^3 - 8x^2 - x + 2$. Try x = 2. $32 - 32 - 2 + 2 = 0$, so we know $(x - 2)$ divides the polynomial.

$$\begin{array}{r|l} & 4x^2 \qquad - 1 \\ \hline x-2 & 4x^3 - 8x^2 - x + 2 \\ & -4x^3 + 8x^2 \\ \hline & \qquad -x + 2 \\ & \qquad\ \ x - 2 \\ \hline & \qquad\quad 0 \end{array}$$

We are left with $4x^2 - 1$, which factors into $(2x - 1)(2x + 1)$, so our answer is $(x - 2)(2x - 1)(2x + 1)$.

Problem 1.69

If we graph $f(x) = 4ln(x) - x$, we can see the roots are between 1 and 2 and 8 and 9. Let's start with the root between 1 and 2.

$f(1) \cong -1$	Lower endpoint is negative.
$f(2) \cong 0.772588722$	Upper endpoint is positive.
$f(1.5) \cong 0.121860432$	Positive: $1 \leq x \leq 1.5$
$f(1.25) \cong 0.357425795$	Negative: $1.25 \leq x \leq 1.5$
$f(1.375) \cong 0.101185076$	Negative: $1.375 \leq x \leq 1.5$
$f(1.4375) \cong 0.014121975$	Positive: $1.375 \leq x \leq 1.4375$
$f(1.40625) \cong 0.042543652$	Negative: $1.40625 \leq x \leq 1.4375$
$f(1.421875) \cong 0.013969307$	Negative: $1.421875 \leq x \leq 1.4375$
$f(1.4296875) \cong 0.000136056$	Answer.

So our first answer is 1.429. Let's look at the second root:

$f(9) \cong 0.211101691$	Lower endpoint is negative.
$f(10) \cong 0.772588722$	Upper endpoint is positive.
$f(1.5) \cong 0.121860432$	Positive: $1 \leq x \leq 1.5$
$f(1.25) \cong 0.357425795$	Negative: $1.25 \leq x \leq 1.5$
$f(1.375) \cong 0.101185076$	Negative: $1.375 \leq x \leq 1.5$
$f(1.4375) \cong 0.014121975$	Positive: $1.375 \leq x \leq 1.4375$
$f(1.40625) \cong 0.042543652$	Negative: $1.40625 \leq x \leq 1.4375$
$f(1.421875) \cong 0.013969307$	Negative: $1.421875 \leq x \leq 1.4375$
$f(1.4296875) \cong 0.000136056$	Answer

Problem 1.71

If we graph $x^3 + x^2 + 2x - 6 = 0$, we can see the root is between 1 and 2.

$f(1) \cong -2$	Lower endpoint is negative.
$f(2) \cong 8$	Upper endpoint is positive.
$f(1.5) \cong 2.625$	Positive: $1 \leq x \leq 1.5$
$f(1.25) \cong 0.015625$	Positive: $1 \leq x \leq 1.25$
$f(1.125) \cong 1.060546875$	Negative: $1.125 \leq x \leq 1.25$
$f(1.1875) \cong 0.540283203$	Negative: $1.1875 \leq x \leq 1.25$
$f(1.21875) \cong 0.266876221$	Negative: $1.21875 \leq x \leq 1.25$
$f(1.234375) \cong 0.126773834$	Negative: $1.234375 \leq x \leq 1.25$
$f(1.2421875) \cong 0.055862904$	Negative: $1.2421875 \leq x \leq 1.25$
$f(1.24609375) \cong 0.020191252$	Negative: $1.24609375 \leq x \leq 1.25$
$f(1.248046875) \cong 0.002301224$	Negative: $1.248046875 \leq x \leq 1.25$
$f(1.249023437) \cong 0.006657356$	Answer

So our solution is approximately 1.249.

A.2 Selected Answers from Chapter 2

Problem 2.1

a) x = $\frac{2}{5}$, y = $\frac{4}{5}$. **c)** Infinite solutions. **e)** x = $\frac{-1}{2}$, y = 3, z = $\frac{-1}{2}$. **g)** x = $\frac{28}{9}$, y = $\frac{173}{54}$, z = $\frac{5}{6}$. **i)** Inconsistent system.

Problem 2.2

x = 22250, y = 45500, z = -12750

Problem 2.3

a) a = 0, b = $\frac{2}{3}$, c = $\frac{1}{3}$, d = 1.

c) a = -1.96, b = -0.978, c = 0.593, d = -0.375.

Problem 2.4

a) C = 1 for infinite solutions, C$\neq$ 1 for inconsistent.

c) A, B, C = 1 gives infinite solutions, E$\neq$ 2 causes the system to be inconsistent.

Problem 2.5
Adults = 200, Children = 420

Problem 2.6
35

Problem 2.7
$-5x^2 + 12x + 4$

Problem 2.8
Bushels cost 23, boxes cost 47.

Problem 2.9
Boat speed is 45, river speed is 15.

Problem 2.10
12 shots worth 2 points and 26 shots worth 3 points were scored.

Problem 2.11

$x^2 + y^2 + 12.967x + 8.567y - 55.867 = 0$

Problem 2.12

37.455 gallons of the first solution, 56 gallons of the second solution and 6.545 gallons of the third solution were used.

A.3 Selected Answers from Chapter 3

Problem 3.1

a) 2, 5, 10, 17, 26, 37.

c) 0, 1, 5, 19, 65, 211.

e) $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$

Problem 3.3
2.1 **a)** finite, **c)** finite, **e)** infinite
2.2 **a)** finite, **c)** infinite, **e)** infinite

Problem 3.5

a) 1, 3, 6, 10, 15, 21.

c) 1, -1, 2, -2, 3, -3.

e) $1, 1+\frac{3}{2}=\frac{5}{2}, 1+\frac{3}{2}+\frac{9}{4}=\frac{19}{4}, 1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}=\frac{65}{8}, 1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}=\frac{211}{16}, 1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\frac{243}{32}=\frac{665}{32}$.

Problem 3.7

a) $\sum_{n=1}^{\infty} \frac{3}{2^n} = 3 \times \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 3 \times 1 = 3$.

c) $\sum_{n=0}^{\infty} \frac{7n}{n+1} = 7 \times \sum_{n=0}^{\infty} \frac{n}{n+1} = 7 \times \infty = \infty$.

e) $\sum_{n=1}^{\infty} \frac{3n}{n+1} - \frac{1}{2^n} = 3 \times \sum_{n=1}^{\infty} \frac{n}{n+1} - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 3 \times \infty - 1 = \infty$

Problem 3.9

a) $\frac{2}{n} > \frac{1}{n}$, for $n \geq 1$, this series diverges.

c) $\frac{1}{3n^2} < \frac{1}{n^2}$, converges by the p-series property.

e) $\frac{1}{n^2+n} < \frac{1}{n^2}$, converges by the p-series property.

Problem 3.11

We can rewrite this limit as $2 \times \lim_{n\to\infty} \frac{n^2}{n^2+1}$. Now let's consider the limit. If we write out the first few terms of the series, we get $\left(0, \frac{1}{2}, \frac{4}{5}, \frac{9}{10}, ...\right)$. As we can see, the numbers get closer and closer to 1 without ever equaling it, so our limit is 1. Then multiplying the limit by the constant, we can see that the total limit is 2.

Problem 3.13

Let's consider the first few terms of the sequence, $\left(\frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \frac{25}{6} ...\right)$. Clearly, the terms are getting bigger. Since we have no upper bound on our limit, this means that the sequence cannot possibly converge. Therefore, it must diverge.

Problem 3.15

Consider the sequence of partial sums of $\sum_{n=k}^{\infty} a_n$. If it doesn't converge to zero, then adding all the terms from $a_0, ..., a_{k-1}$ won't make it converge to zero either. If the entire sequence of partial sums also doesn't converge to zero, then the part of the sequence that starts when $n = k$ won't converge to zero either. Thus, if one diverges, they both diverge.

Now assume the series $\sum_{n=0}^{\infty} a_n$ does converge. That means that it has a sum L. Then this implies that if we start the series at $n = k$, then the sum will be $L - (a_0 + ... + a_{k-1})$, which means that both are convergent sums. Using reverse reasoning, let $\sum_{n=k}^{\infty} a_n$ converge to a sum L. Since there can only be a finite number of terms before $n = k$, they must have a finite sum. Thus if one series converges, both have to converge.

Problem 3.19

The rule is to find the next number is add the 3 numbers that come before it, so the next 3 numbers are: 105, 193, 355.

Problem 3.21

This numbers are generated by the hailstone problem. If the number is odd, multiply it by 3 and add 1 to get the next number. If its even, divide it by 2 to get the next number.

Problem 3.23

a) Geometric, it has a ratio of $\frac{3}{4}$. **c)** Not geometric, it does not have a constant ratio. **e)** Geometric, has a constant ratio of $\frac{2}{3}$.

Problem 3.25

a) $2 \times \left(\frac{1}{1-\frac{1}{3}}\right) = 3$

c) $7 \times \left(\frac{1}{1-\frac{1}{3}}\right) = \frac{21}{2}$

e) $\left(\frac{1}{1-\frac{2}{5}}\right) = \frac{5}{3}$

g) Divergent series, undefined.

i) $3 \times \frac{1}{1-(-0.9)} = \frac{30}{19}$

Problem 3.27

$\sum_{n=0}^{\infty} \left(\frac{1}{5}\right) \times \left(\frac{4}{5}\right)^n$

Problem 3.29

There are two series to build in this solution. The first is for how far the ball drops each time. The first few terms are $1 + 0.9 + (0.9)^2 + (0.9)^3 + \ldots$, which can be written as $\sum_{n=0}^{\infty}(0.9)^n$. The second series is how far the ball travels each bounce up, which is $0.9 + (0.9)^2 + (0.9)^3 + (0.9)^4 + \ldots$, so we can write this sequence as $\sum_{n=0}^{\infty}(0.9)(0.9)^n$. Adding both of these series up, it becomes:

$$\frac{1}{1-0.9} + \frac{0.9}{1-0.9} = \frac{1}{0.1} + \frac{0.9}{0.1} = 10 + 9 = 19$$

Problem 3.31

True. This first term, 12 in this case, vanishes after a while and what you are left with is a series with the first term of 3 and a ratio of $\frac{1}{2}$.

Problem 3.33

$$10 + 10(0.875) + 10(0.875)^2 + \ldots + 10(0.875)^n + \ldots = \sum_{n=0}^{\infty} 10 \times (0.875)^n$$
$$= \frac{10}{1 - 0.875} = \frac{10}{\frac{1}{8}} = 80$$

Problem 3.35

Let's calculate the answer by finding out how much string we will be cutting away as we reach infinity (assuming we could get there). In the first round, we cut away $\frac{1}{4}$, leaving $\frac{3}{4}$ total. Since there are two pieces, each piece is $\frac{3}{8}$ long. Now we cut away a quarter of each string, so $\frac{1}{4} \times \frac{3}{8} = \frac{3}{32} \times 2 = \frac{3}{16}$, is cut away. Since there was $\frac{12}{16}$ of the original string left, cutting away $\frac{3}{16}$ leaves us with $\frac{9}{16}$. If we continue on that path, the terms of our series will be $(\frac{1}{4}, \frac{3}{16}, \frac{9}{64}, \ldots)$. Now we can sum up how much string we are cutting away:

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^n = \frac{1}{4} \times \frac{1}{1 - \frac{3}{4}} = \frac{1}{4} \times 4 = 1$$

Which means that we are cutting away everything, since 1 - 1 = 0. Thus, there is nothing left of the string.

Problem 3.37

Before we make our table, we should find out what interest we will be compounding with. Since it's quarterly, we have to divide the annual interest by 4, $\frac{0.04}{4} = 0.01$. Now we can make our table.

Month	Balance	Interest	Deposit
1st Q	0.00	0.00	50000.00
2nd Q	50000.00	500.00	50000.00
3rd Q	100500.00	1005.00	50000.00
4th Q	151505.00	1515.05	50000.00
5th Q	203020.05	2030.20	50000.00
6th Q	255050.25	2550.50	50000.00
7th Q	307600.75	3076.01	50000.00
8th Q	360676.76	3606.77	50000.00
9th Q	414283.53	4142.84	50000.00
10th Q	468426.37	4684.26	50000.00
11th Q	523110.63	5231.11	50000.00
12th Q	528341.74	5283.42	50000.00
Final balance		**583625.16**	

Problem 3.39

1852.70 For the table method but 1852.71 by using the formula. In this instance, rounding yields a one cent difference.

Problem 3.41

a) $50 \times \left(\frac{(1.0025)^{60}-1}{0.0025}\right) = 3232.34$

c) $100 \times \left(\frac{(1+0.02/12)^{120}-1}{0.02/12}\right) = 13271.97$

e) $50 \times \left(\frac{(1+0.04/12)^{240}-1}{0.04/12}\right) = 18338.73$

Problem 3.43

a) $\frac{5000(1.0033)^{60}(.0033)}{1.0033^{60}-1} = 91.99$

c) $\frac{220000(1.0033)^{240}(.0033)}{1.0033^{240}-1} = 7324.04$

e) $\frac{200(1.0125)^{24}(.01215)}{1.0125^{24}-1} = 9.70$

Problem 3.45

a) $\log_{1.0033}\left(\frac{400}{400-5000(0.0033)}\right) = \frac{log(1.043024772)}{log(1.0033)} = 12.786180127 =\cong 13$

c) $\log_{1.0025}\left(\frac{2600}{2600-1000000(0.0025)}\right) = \frac{log(26)}{log(1.0025)} = 1304.866985555 \cong 1305$

e) $\log_{1.0033}\left(\frac{10}{10-2000(0.0033)}\right) = \frac{log(2.941176471)}{log(1.0033)} = 327.451127287 \cong 328$

Problem 3.47

Using the formula from Fact 3.9, we can rearrange it to find P.

$$Pm^c = p \cdot \frac{m^c - 1}{m - 1} \rightarrow P = \frac{p}{m^c} \cdot \frac{m^c - 1}{m - 1}$$

Knowing that $m = 1.0033$, $c = 217$, and $p = 40$, we plug these values into the formula.

$$P = \frac{40}{1.0033^{217}} \cdot \frac{(1.0033)^{217} - 1}{0.0033} = 6191.14$$

Problem 3.49

$$p = \frac{Pm^c(m-1)}{m^c - 1} = \frac{1200(1.00083)^{24}(0.00083)}{(1.00083)^{24} - 1} = 50.52$$

Problem 3.51

We need to find out what the balance will be at the end of the two years first, then we can calculate what payments will be needed to pay it off in one year.

$$1200(1.00083)^{24} = 1224.13$$

Now we can use the formula to find the amount each payment will have to be per month in one year.

$$p = \frac{1224.13(1.00083)^{12}(0.00083)}{(1.00083)^{12} - 1} = 102.56$$

Problem 3.53

Without the preprocessing fee, $100000(1.003125)^{36} = 111887.60$. With, $98800(1.002708333)^{36} = 108903.90$. Clearly, paying the preprocessing fee is the smarter choice, with a difference of 111887.60 - 108903.90 = 2983.70.

Problem 3.57

$$B = \left(1 + \frac{i}{100}\right) D \cdot \frac{(1 + i/100)^c - 1}{i/100}$$

Problem 3.59

We need to calculate what the daily interest is going to be. Assuming that a month has 30 days (we could assume 31, the way we get the answer is the same), then 10% monthly interest becomes $\frac{0.1}{30} = 0.0033$, which we can now use in our compound interest formula:

$$200(1.0033)^{14} = 209.44$$

A.4 Selected Answers from Chapter 4

Problem 4.1

a) Limit exists. **c)** Limit exists. **e)** Limit exists.

Problem 4.3

a) Has a limit = 1.

c) Limit does not exist.

e) Has a limit = 1.

Problem 4.5

a) $\underset{x\to\infty}{Lim}\frac{1}{x^2} = 0$, since the bottom is continually getting larger while the top remains constant.

c) $\underset{x\to\infty}{Lim}\frac{x^2+x+1}{1-3x+x^2} = 1$. The numerator has x^2 as its highest power of x, and the denominator has the same. Thus, when taking the limit as $x \to \infty$, all other terms will cancel out, leaving $\frac{x^2}{x^2} = 1$

e) $\underset{x\to\infty}{Lim}\frac{x^3-1}{x^2+1} = \infty$. This expression has no limit, since the degree of the top is higher than the degree of the bottom, and both the numerator and denominator have positive coefficients for the highest powers of x.

Problem 4.7

a) $\underset{x\to a}{Lim}(f(x)+h(x)) = \underset{x\to a}{Lim}f(x) + \underset{x\to a}{Lim}h(x) = 3-2 = 1.$

c) $\underset{x\to a}{Lim}\left(\frac{f(x)}{g(x)}\right) = \frac{3}{0} =$ Undefined.

e) $\underset{x\to a}{Lim}(f(x)-h(x))^2 = (3-(-2))^2 = 5^2 = 25.$

Problem 4.9
Consider x^2-4. This is a factorable polynomial, and using difference of squares we can get $x^2-4 = (x+2)(x-2)$. Using this fact, let's reconsider the limit.

$$\underset{x\to 2}{Lim}\frac{x^2-4}{x-2} = \underset{x\to 2}{Lim}\frac{(x+2)(x-2)}{x-2} = \underset{x\to 2}{Lim}x+2 = 4$$

.
Problem 4.11
Using the speed of growth hierarchy we know polynomials grow faster than logarithmics,

$$\underset{x\to\infty}{Lim}\frac{log(x^5)}{x+5} = 0$$

.
Problem 4.13

$$\underset{x\to\infty}{Lim}\frac{e^x+1}{ln(x)+1} = \infty$$

.
Problem 4.15
We need to find when these two equations are equal when $x = a$, so set them equal to each other, shift everything to one side, and solve for a.

$$a^2 = 2a+3 \to a^2-2x-3 = 0 \to (a-3)(a+1) = 0$$

This function is continuous when $a = 3, -1$.

Problem 4.17
Similar steps as problem 3.15.

$$a^3 = 4x \to a^3 - 4x = 0 \to a(a^2 - 4) = a(a-2)(a+2) = 0$$

This function is continuous when $a = 0, 2, -2$.

Problem 4.22
Use the limit based definition of the derivative to compute the derivative of each of the following functions. The algebra on the last two functions can get a bit intense.

a) $f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \lim_{h\to 0} \frac{2xh+h^2}{h} = 2x.$

c)

$$\begin{aligned}
f'(x) &= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}}{h} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{\frac{-2xh-h^2}{x^2(x+h)^2}}{h} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{h \cdot \frac{-2x-h}{x^2(x+h)^2}}{h} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{\not{h} \cdot \frac{-2x-h}{x^2(x+h)^2}}{\not{h}} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{-2x-h}{x^2(x+h)^2} \\
&= \frac{-2x}{x^4} \\
&= \frac{-2}{x^3}
\end{aligned}$$

e)

$$
\begin{aligned}
f'(x) &= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{\frac{1}{(x+h)^2+1} - \frac{1}{x^2+1}}{h} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{\frac{x^2+1-((x+h)^2+1)}{(x^2+1)((x+h)^2+1)}}{h} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{x^2+1-(x^2+2xh+h^2+1)}{h(x^2+1)((x+h)^2+1)} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{-2xh-h^2}{h(x^2+1)((x+h)^2+1)} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{-2x-h}{(x^2+1)((x+h)^2+1)} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{-2x}{(x^2+1)(x^2+1)} \\
&= \begin{matrix} Lim \\ h \to 0 \end{matrix} \frac{-2x}{(x^2+1)^2}
\end{aligned}
$$

Problem 4.24
For each of the following functions, find the derivative using the product rule.

a) $f'(x) = (2x+1)(x^2-3x-2)+(x^2+x+1)(2x-3)$ **c)** $f'(x) = e^x + xe^x$

e) $f'(x) = 2xln(x) + x$

Problem 4.26
For each of the following functions, find the derivative using the quotient rule.

a) $f'(x) = \frac{1-x^2}{(x^2+1)^2}$ **c)** $f'(x) = \frac{3x^2-4x}{(3x-2)^2}$

e) $f'(x) = \frac{2e^x}{(e^x+2)^2}$

Problem 4.28
Find the tangent line at the indicated value of x.

a) $y = 1$.

c) $y = -2x + 7$.

e) $y = \frac{-5}{4}x + \frac{11}{4}$.

Problem 4.30
First, we need to find when $x^3 - 4x$ is equal to $y = 0$. So, $x^3 - 4x = x$ and then $x^3 - 5x = 0 \to x(x^2-5) = 0$. We know then that $x = 0, \sqrt{5}, -\sqrt{5}$. If we plug any values into either function, we respectively get $y = 0, \sqrt{5}, -\sqrt{5}$. Now we take the derivative of $f(x)$, so $f'(x) = 3x^2 - 4$ and plug in the values of x we found earlier to get the slopes of the tangent lines. So if we plug in $x = 0, \sqrt{5}, -\sqrt{5}$, we get $f'(x) = -4, 11, 11$, respectively. Then,

$$y - 0 = -4(x-0) \to y = -4x$$

$$y - \sqrt{5} = 1(x - \sqrt{5}) \to y = 11x - 10\sqrt{5}$$

$$y + \sqrt{5} = 1(x + \sqrt{5}) \to y = 11x + 10\sqrt{5}$$

Problem 4.32
Find the derivative first and set it equal to 0, $f'(x) = \frac{-6x+12}{(x^2-4x+5)^2} = 0$. We only need to solve for the top, which gives us $x = 2$. Make sure to check that the point doesn't cause a division by 0. Plug our x value into the y, which gives us $y = 3$, our solution.

Problem 4.34

$$y' = \frac{3(e^{2x}+5)^2 \cdot (2e^{2x})(e^{2x}+1) - (e^{2x}+5)^3 \cdot (2e^{2x})}{(e^{2x}+1)^2}$$

$$= \frac{4e^{2x}(e^{2x}+5)^2(e^{2x}-1)}{(e^{2x}+1)^2}$$

Problem 4.36

$$P'(x) = \frac{(14x^3+16.2x)(x^3+3x+1) - (3.5x^4+8.1x^2)(3x^2+3)}{(x^3+3x+1)^2}$$

$$P'(4) = 3.66$$

$$P'(20) = 3.51$$

$$P'(50) = 3.50$$

A.5 Selected Answers from Chapter 5

Problem 5.1

a) $f(x) = x^2 - 2x - 3 = (x-3)(x+1)$ Roots at x = 3, -1.

c) $f(x) = x^3 - 5x^2 + 7x - 2$. We check for roots that will make $f(x) = 0$, in this case x = 2. Then,

$$
\begin{array}{r}
x^2 - 3x + 1 \\
x-2 \overline{)\, x^3 - 5x^2 + 7x - 2} \\
\underline{-x^3 + 2x^2 } \\
-3x^2 + 7x \\
\underline{3x^2 - 6x } \\
x - 2 \\
\underline{-x + 2} \\
0
\end{array}
$$

leaves us with $x^2 - 3x + 1 \rightarrow x = \frac{3 \pm \sqrt{5}}{2}$ which are our 2 other roots.

e) $f(x) = \frac{x^2+x-2}{x^2+1}$. We only need to consider the top when dealing with roots, so $x^2 + x - 2 = (x+2)(x-1)$. Roots at x = -2, 1

g) $f(x) = \sqrt{x^3 - 2x^2 - 5x + 6}$. Try possible roots, see that x = 1 works. Then,

$$
\begin{array}{r}
x^2 - x - 6 \\
x-1 \overline{)\, x^3 - 2x^2 - 5x + 6} \\
\underline{-x^3 + x^2 } \\
-x^2 - 5x \\
\underline{x^2 - x } \\
-6x + 6 \\
\underline{6x - 6} \\
0
\end{array}
$$

leaves us with $x^2 - x - 6 = (x-3)(x+2)$, roots at x = -2, 3.

i) $f(x) = e^{x^2-3x+2}$. No roots, $e^x > 0$ for all x.

Problem 5.3

a) $-1 \leq x \leq 5$	$[-1, 5]$	The set of all numbers between -1 and 5, inclusive.
c) $2 \leq x < 7$	$[2, 7)$	The set of all numbers greater than or equal to two and strictly less than 7.
e) $-\infty < x < \infty$	$(-\infty, \infty)$	The set of all real numbers

Problem 5.5

a) Decreasing for $(-\infty, -1.5)$, increasing on $(-1.5, \infty)$.

c) Decreasing for $(-2, 2)$, increasing on $(-\infty, -2) \bigcup (2, \infty)$.

e) Increasing $(-\infty, \infty)$.

Problem 5.8

a)

Summary table

Function :	$f(x) = x^3 - 3x + 2$
Roots :	$x = 1, -2$
Vertical asymptotes :	none.
Horizontal asymptotes :	none.
Critical points :	$x = \pm 1$
Increasing on :	$(-\infty, -1) \cup (1, \infty)$
Decreasing on :	$(-1, 1)$
Inflection points :	$x = 0$
CCU on :	$(0, \infty)$
CCD on :	$(-\infty, 0)$

c)

Summary table

Function :	$f(x) = \frac{x}{(x-1)^2}$
Roots :	$x = 0$
Vertical asymptotes :	$x = 1$.
Horizontal asymptotes :	$y = 0$.
Critical points :	$x = \pm 1$
Increasing on :	$(-\infty, -1) \cup (1, \infty)$
Decreasing on :	$(-1, 1)$
Inflection points :	$x = 0$
CCU on :	$(1, \infty)$
CCD on :	$(-\infty, 1)$

e)

Summary table

Function :	$f(x) = x^2 e^{-x}$
Roots :	$x = 0$
Vertical asymptotes :	none.
Horizontal asymptotes :	$y = 0$.
Critical points :	$x = 0, 2$
Increasing on :	$(0, 2)$
Decreasing on :	$(-\infty, 0) \bigcup (2, \infty)$
Inflection points :	$x = 0$
CCU on :	$(-\infty, 2 - \sqrt{2}) \bigcup (2 + \sqrt{2}, \infty)$
CCD on :	$(2 - \sqrt{2}, 2 + \sqrt{2})$

Problem 5.11

Summary table

Function :	$f(x) = \frac{x^2+x-8}{x-2}$
Roots :	no roots
Vertical asymptotes :	$x = 2$
Horizontal asymptotes :	none.
Diagonal asymptotes :	$y = x$
Critical points :	$x = 2$
Increasing on :	$(-\infty, 2) \cup (2, \infty)$
Decreasing on :	none.
Inflection points :	$x = 0$
CCU on :	$(-\infty, 2) \cup (2, \infty)$
CCD on :	none.

Problem 5.13

$f^{(3)}(x) = 3e^x + xe^x$

Problem 5.15

Try

$$f(x) = \begin{cases} x & x < 0 \\ 0 & 0 \leq x \leq 1 \\ x-1 & x > 1 \end{cases}$$

for example.

Problem 5.23

a) $f(x) = x^2 + 3x + 5$. No horizontal asymptotes.

c) $f(x) = x^4 - 32x^2 + 5$. No horizontal asymptotes.

e) $f(x) = (2x+1)e^{-x}$. $y = 0$.

g) $f(x) = 4x + \frac{10}{x}$. No horizontal asymptotes.

Problem 5.25

a) $x = \frac{-3}{2}$ global minimum.

c) $x = 0, \pm 4$ 0 is a local maximum, ± 4 are global minima.

e) $x = \frac{1}{2}$. no classification.

g) $x = 0, \pm\sqrt{\frac{5}{2}}$. 0 has no classification, $\sqrt{\frac{5}{2}}$ are global maxima.

Problem 5.27

a) $n = 42$.

c) $n = 8$.

e) $n = 7$.

Problem 5.29

$12x + 7y = P$, $xy = 100$

$$x = \frac{100}{y}$$

$$\frac{1200}{y} + 7y = P$$

$$0 = \frac{14y^2 - 1200 - 7y^2}{y^2}$$

$$0 = 7y^2 - 1200$$

$$y = \pm\sqrt{\frac{1200}{7}} \cong 13.07$$

$$13.07x = 100 \rightarrow x \cong 7.64$$

Problem 5.31

$V = 1600 = \pi \cdot r^2 \cdot h$, $A = \pi \cdot r^2 + 2\pi \cdot r \cdot h$.

$$h = \frac{1600}{\pi \cdot r^2}$$

$$A = \pi \cdot r^2 + \frac{3200}{r}$$

$$0 = 2\pi \cdot r - \frac{3200}{r^2}$$

$$2\pi \cdot r^3 = 3200$$

$$r = 7.98589$$

$$h = \frac{1600}{\pi \cdot (7.98589)^2} = 7.98589$$

Problem 5.33

$40W + 120L = P$, $LW = 8000$

$$W = \frac{8000}{L}$$

$$\frac{320000}{L} + 120L = A$$

$$\frac{-320000}{L^2} + 120 = 0$$

$$L^2 = \frac{320000}{120}$$

$$L \cong 51.64$$

$$W = \frac{8000}{51.64} = 154.92$$

Problem 5.36

a) 900 m^2

b) Yes.

c) $y \cong 51.09$

d) 60.

A.6 Selected Answers from Chapter 6

Problem 6.1

a) $F(x) = \frac{x^6}{6} + C$.

c) $F(x) = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} - 7x + C$.

e) $F(x) = \int ((x-1)(x-2)(x-3)) \cdot dx = \int x^3 + 6x^2 + 11x - 6 \cdot dx = \frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x + C$.

Problem 6.3

a) $F(x) = \frac{(x^2-3)^3}{3} + C$. **c)** $F(x) = \frac{1}{6} \cdot ln(2x^3 - 4) + C$.

e) Hint, let $u = ln(x)$. $F(x) = ln(ln(x))$.

Problem 6.5

$$\int \frac{e^x}{(e^x+4)^3} = \frac{-1}{(e^x+4)^{-2})} = -(e^x+4)^{-2} + C$$

Problem 6.7

Compute

$$\int_0^{ln(5)} e^{-x} \cdot dx = -e^{-x}\Big|_0^{ln5} = -e^{-ln5} + e^0 = 5 + 1 = 6$$

Problem 6.9

$$\int_1^3 \frac{1}{x \cdot ln(x)} \cdot dx = ln(ln(x))|_1^3 = ln(ln(3)) - ln(ln(1))$$

We have a problem here, this gives us a ln(0) in the second term...

Problem 6.11 Let $u = cx + d$. Then solve for x, $x = \frac{u-d}{c}$, and find dx, $dx = \frac{du}{c}$. Now substitute:

$$\int \frac{a \cdot \frac{u-d}{c} + b}{u} \cdot \frac{du}{c} = \frac{1}{c} \int \frac{au}{cu} - \frac{ad}{cu} + \frac{b}{u} \cdot du = \frac{a}{c^2} \cdot u - \frac{ad}{c^2} \cdot ln(u) + \frac{b}{c} \cdot lnu + C$$

Problem 6.15

Let $c^x = e^{xln(c)}$, then let $u = xln(c)$, which means $du = ln(c) \cdot dx \rightarrow dx = \frac{du}{ln(c)}$. Now substitute:

$$\int e^u \cdot \frac{du}{ln(c)} = \frac{e^u}{ln(c)} = \frac{e^{xln(c)}}{ln(c)} = \frac{c^x}{ln(c)} + C$$

Problem 6.21

a) $F(x) = -(x^3 + 3x^2 + 6x + 6) \cdot e^{-x}$. **c)** $f(x) = \int \left(x^2 + 4x + 8\right) \cdot e^{-x}$.

e) $f(x) = \int \left(x^3 + 4x^2 + 9x + 10\right) \cdot e^{-x}$.

Problem 6.23

$$\int_0^{12} 1.1 + 0.6t \cdot dt = 1.1t + 0.3t^2\Big|_0^{12} = 56.4$$

Problem 6.25

$$\int_0^{52} \frac{10w}{10w^2+1} \cdot dw = \frac{1}{2} \cdot ln(10w^2+1)\Big|_0^{52} = \frac{1}{2} \cdot ln(52^2+1) \cong 3.95$$

Problem 6.27

$$500 = \int_0^6 \pi \left(x^2 + q)^2\right) \cdot dx$$

$$\pi \int_0^6 \frac{x^4}{144} + \frac{2x^2q}{12} + q^2 \cdot dx$$

$$\pi \left[\frac{x^5}{720} + \frac{x^3q}{18} + q^2x\right]_0^6$$

$$6q^2 + \frac{6^3q}{18} + \frac{6^5\pi}{720} - 500 = 0$$

$$q^2 + 2q + \frac{6^4\pi}{720} - \frac{500}{6} = 0$$

$$q \cong 7.87$$

Problem 6.29

$$500 = \int_0^8 \pi \left(\frac{x^2}{25} + q)^2\right) \cdot dx$$

$$\pi \int_0^8 \frac{x^4}{625} + \frac{2x^2q}{25} + q^2 \cdot dx$$

$$\pi \left[\frac{x^5}{3125} + \frac{2x^3q}{75} + q^2x\right]_0^8$$

$$8q^2 + 13.6533q - 156.181 = 0$$

$$q \cong 3.65$$

Problem 6.31

Integrate the failure rate of the given type of lighbulbs until the total failure, as computed by the integral, is one half:

$$0.5 = \int_0^H 4e^{-4t} \cdot dt$$

$$0.5 = 1 - e^{-4H}$$

$$\frac{ln(\frac{1}{2})}{-4} = H \to H \cong 0.173 \text{ years}$$